A SIMPLE PROOF OF A COVERING PROPERTY OF LOCALLY COMPACT GROUPS

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ABSTRACT. We give a simple proof of the following result of Emerson and Greenleaf.

THEOREM. Let V be a relatively compact subset with nonvoid interior of a locally compact group G. Then there exist a subset T ⊆ G and a natural number M such that G = \bigcup_{t ∈ T} tV and at most M of the tV's, t ∈ T, intersect.

The result cited above is proved in [4] and is used there and in [2] in the course of proving that every amenable locally compact group G has strong properties such as

(A) If ε > 0 and compact K ⊆ G containing the identity of G are given, there is a compact U ⊆ G with |U| > 0 such that |KU \Delta U|/|U| < ε.

(Here |U| indicates left Haar measure of the set U. And we remind the reader that G is called amenable if L^∞(G) admits a left invariant mean; see [6], [8], [1] for further details.)

The proof given in [4] of the theorem above involves some delicate arguments about geometry of groups. We discovered the simple proof presented below in the course of preparing [1] (and were apprised later that it is almost the same as a proof in Chapter 8, §1.7, of [7]); our reason for publishing it now is that it seems not widely known, according to [3], [5], that such a proof exists.

PROOF OF THE THEOREM. After a reduction as in [2; Proposition 2], we are left with the task of taking a relatively compact symmetric neighbourhood V of e ∈ G and finding T ⊆ G and constant M so that G = \bigcup_{t ∈ T} tV and at most M of the tV's, t ∈ V, intersect. We may assume the open and closed subgroup \bigcup_{t ∈ T} tV of G equals G. (For, if we cover \bigcup_{t ∈ T} tV with \bigcup_{t ∈ T} tV, then \bigcup_{t ∈ T} tV covers the coset s \bigcup_{t ∈ T} tV and hence we cover the whole group.) And we may assume the subgroup \bigcup_{t ∈ T} tV is not compact. (Otherwise we can cover it with a finite number of left translates of V and proceed as in the previous parenthetical remark.) We then get our set T ⊆ G as follows.

Let t_1 = e. Since G is not compact, \bigcup_{t ∈ T} tV is not compact. And there is a t_2 ∈ (V^2)^- \setminus V.

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If \((V^2)^- \setminus \bigcup t_i V \neq \emptyset\), take \(t_3\) in it. Continuing like this, we get 
\[(V^2)^- \subset \bigcup_{i=1}^{N_2} t_i V\]
with
\[j-1\]
\[t_j \notin \bigcup_{i=1}^{j-1} t_i V, \quad 2 \leq j \leq N_2.\]

(Note that, if \(W\) is a symmetric neighbourhood of \(e\) such that \(W^2 \subset V\), then 
\[N_2 \leq \frac{|(V^2)^- W|}{|W|} \cdot |(V^2)^- W|/|W| \cdot |(V^2)^- W|/|W| \ldots \]
choose \(t_{N_2+1}\) in it. And so on. Hence, by induction, we get
\[(V^n)^- \subset \bigcup_{i=1}^{N_n} t_i V\]
with
\[j-1\]
\[t_j \notin \bigcup_{i=1}^{j-1} t_i V, \quad 2 \leq j \leq N_n.\]

thus \(G = \bigcup_{j=1}^{\infty} t_j V\) (and \(N_n \leq \frac{|(V^n)^- W|}{|W|} \cdot |(V^n)^- W|/|W| \ldots \).

Suppose \(s \in t_i V\). Then \(t_i s \in s V W\) and \(t_i W \in s V W\) (where \(W^2 \subset V\) as above). And, if \(s\) is also in \(t_j V\), then 
\(t_j W \subset s V W\) with \(t_j W \cap t_j W = \emptyset\) if \(i \neq j\). It follows that \(s\) is contained in at most \(|V W|/|W|\) of the \(t_i V\)'s, \(i = 1, 2, 3, \ldots\).

References


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