A SIMPLE PROOF OF A COVERING PROPERTY OF LOCALLY COMPACT GROUPS

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ABSTRACT. We give a simple proof of the following result of Emerson and Greenleaf.

THEOREM. Let V be a relatively compact subset with nonvoid interior of a locally compact group G. Then there exist a subset T ⊂ G and a natural number M such that G = \bigcup_{t \in T} tV and at most M of the tV's, t ∈ T, intersect.

The result cited above is proved in [4] and is used there and in [2] in the course of proving that every amenable locally compact group G has strong properties such as

(A) If ε > 0 and compact K ⊂ G containing the identity of G are given, there is a compact U ⊂ G with |U| > 0 such that |KU ∆ U|/|U| < ε.

(Here |U| indicates left Haar measure of the set U. And we remind the reader that G is called amenable if L∞(G) admits a left invariant mean; see [6], [8], [1] for further details.)

The proof given in [4] of the theorem above involves some delicate arguments about geometry of groups. We discovered the simple proof presented below in the course of preparing [1] (and were apprised later that it is almost the same as a proof in Chapter 8, §1.7, of [7]); our reason for publishing it now is that it seems not widely known, according to [3], [5], that such a proof exists.

PROOF OF THE THEOREM. After a reduction as in [2; Proposition 2], we are left with the task of taking a relatively compact symmetric neighbourhood V of e ∈ G and finding T ⊂ G and constant M so that G = \bigcup_{t \in T} tV and at most M of the tV's, t ∈ V, intersect. We may assume the open and closed subgroup \bigcup_{t \in T} \langle t \rangle V of G equals G. (For, if we cover \bigcup_{t \in T} \langle t \rangle V with \bigcup_{t \in T} tV, then \bigcup_{t \in T} \langle t \rangle V covers the coset s \cup \langle t \rangle V and hence we cover the whole group.) And we may assume the subgroup \bigcup_{t \in T} \langle t \rangle V is not compact. (Otherwise we can cover it with a finite number of left translates of V and proceed as in the previous parenthetical remark.) We then get our set T ⊂ G as follows.

Let t_1 = e. Since G is not compact, V^2 ≠ V and there is a t_2 ∈ (V^2)^- \setminus V.

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If \((V^2)^- \setminus \bigcup_j t_j V \neq \emptyset\), take \(t_3\) in it. Continuing like this, we get \((V^2)^- \subseteq \bigcup_j t_j V\) with
\[
j-1\sum_{j=1}^{n_2} t_j \subseteq \bigcup_j t_j V, \quad 2 \leq j \leq n_2.
\]
(Note that, if \(W\) is a symmetric neighbourhood of \(e\) such that \(W^2 \subseteq V\), then \(n_2 < \frac{|(V^2)^- W|}{|W|}\).) If \((V^3)^- \setminus \bigcup_j t_j V \neq \emptyset\), choose \(t_{n_2+1}\) in it. And so on.

Hence, by induction, we get \((V^n)^- \subseteq \bigcup_j t_j V\) with
\[
j-1\sum_{j=1}^{n_2} t_j \subseteq \bigcup_j t_j V, \quad 2 \leq j \leq n_n;
\]
thus \(G = \bigcup_j t_j V\) (and \(n_n \leq \frac{|(V^n)^- W|}{|W|}\)). Suppose \(s \in t_j V\). Then \(t_j sVW \subseteq sVW\) (where \(W^2 \subseteq V\) as above). And, if \(s\) is also in \(t_j V\), then \(t_j sVW \subseteq sVW\) with \(t_j sVW \cap t_j sVW = \emptyset\) if \(i \neq j\). It follows that \(s\) is contained in at most \(|VW||W|\) of the \(t_j V\)'s, \(i = 1, 2, 3, \ldots\).

REFERENCES


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