A COUNTEREXAMPLE IN $l_2$-MANIFOLD THEORY

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Abstract. There is a space $X$ such that $X \times X$ is $l_2$, but $X$ is not $l_2$.

In [1], one asks whether $X \times X \approx l_2$ implies $X \approx l_2$. In this note, a counterexample to that question is given.

Let $K$ be a noncontractible, compact homology cell. Then $Y = K \times l_2$ is a noncontractible $l_2$-manifold [4]. Define $X = \text{cone}(Y)$ with metric topology, and $c$ its cone point $\{Y \times 0\}$. If $\alpha$ denotes the ordered pair $(c, c)$ in $X \times X$, then $X - \{c\}$ and $X \times X - \{\alpha\}$ are $l_2$-manifolds. But, $X$ is not since $\{c\}$ is not a $Z$-set in $X$. However,

**Theorem.** $X \times X \approx l_2$.

**Proof.** By Torunczyk's theorem [1, p. 127], it follows that the AR $X \times X$ is an $l_2$-manifold, since $\{\alpha\}$ is a $Z$-set in $X \times X$ by Lemma 2 below. Hence, $X \times X \approx l_2$, by Corollary 3 of [2]. □

In the following, standard notions in infinite dimensional topology as $Z$-set, $l_2 \cong \mathbb{R}^\infty - 1, 1[,$ etc. . . . is referred to Chapman [1].

**Lemma 1.** $X \times X - \{\alpha\}$ is homeomorphic to $l_2$.

**Proof.** By Corollary 3 in [2], Lemma 1 will be proved by showing that $X \times X - \{\alpha\}$ is contractible. We intend to use Whitehead's theorem (Theorem 7.6.25 of [3]). So, two following properties need be verified.

(i) $\tilde{H}_*(X \times X - \{\alpha\}) = 0$.

Let $A = (X - \{c\}) \times X$ and $B = X \times (X - \{c\})$. Then, $A \cap B = (X - \{c\}) \times (X - \{c\})$ and $A \cup B = X \times X - \{\alpha\}$. Since $\tilde{H}_*(X - \{c\}) = \tilde{H}_*(K)$ is trivial, the Künneth formula [3, p. 227] shows that $\tilde{H}_*(A)$, $\tilde{H}_*(B)$ and $\tilde{H}_*(A \cap B)$ are trivial. Finally the Mayer-Vietoris sequence [3, p. 186], for open sets $A$ and $B$ of $X \times X - \{\alpha\}$, shows that $\tilde{H}_*(X \times X - \{\alpha\}) = 0$ as we desired.

(ii) $X \times X - \{\alpha\}$ is simply connected.

Given a map $f: S^1 \to X \times X - \{\alpha\}$, it suffices to show that $f$ is nullhomotopic in $X \times X - \{\alpha\}$. Let $f(t) = (h(t), g(t))$, where $h, g: S^1 \to X$ such that the compact subsets $h^{-1}(c)$ and $g^{-1}(c)$ of $S^1$ are disjoint.

Let $\{ C_j \}_{j = 1, \ldots, m}$ be a finite family of pairwise disjoint arcs of $S^1$ missing $g^{-1}(c)$ such that the union of their interiors covers $h^{-1}(c)$.
A null-homotopy of $f$ in $X \times X - \{\alpha\}$ is the successive composition of three homotopies:

a. homotopes $h$ (rel $S^1 - \bigcup_{j=1}^m \text{Int} C_j$) in $X$ to a map $h_{1/2}$ such that $h_{1/2}(c) = \emptyset$.

b. homotopes $g$ in $X$ to a constant map $g_1$ in $X - \{c\}$.

c. homotopes $h_{1/2}$ in $X$ to a constant map $g_1$ in $X - \{c\}$.

Now, $f = (h, g) \sim (h_{1/2}, g) \sim (h_{1/2}, g_1) \sim (h_1, g_1)$, where $\sim$ means homotopic to. The proof of Lemma 1 is complete.

**Lemma 2.** $\{\alpha\}$ is a $Z$-set in $X \times X$.

**Proof.** Given an open cover $\Omega = \{w_i\}$ of $X \times X$, it suffices to show that there is a map

$$f: X \times X \to X \times X - \{\alpha\}$$

which is $\Omega$-close to $\text{id}_{X \times X}$.

Let $Y_s = Y \times s \subset X$ where $0 < s < 1$ and $V_s = \text{cone}(Y_s) = \{(y, t) \in X | t < s\}$. Then, there is an $s$ ($0 < s < 1$) such that the neighborhood $V_s \times V_s$ of $\alpha$ in $X \times X$ is contained in $w_i \in \Omega$.

It is obvious that $Z = (V_s \times Y_s) \cup (Y_s \times V_s)$ is the topological frontier of $V_s \times V_s$ in $X \times X$ and $V_s \times V_s - \{\alpha\} \approx X \times X - \{\alpha\}$ is an AR. Hence, the inclusion map $Z \to V_s \times V_s - \{\alpha\}$ extends to a map $\varphi: V_s \times V_s \to V_s \times V_s - \{\alpha\}$.

Now, we extend $\varphi$ by the identity map over $X \times X - V_s \times V_s$ to obtain a desired $\Omega$-approximation $f$ of $\text{id}_{X \times X}$. □

**Remark.** Replacing $l_2$ by $Q = \prod_i \mathbb{I}[-1, 1]$ in the above construction, we also obtain a compact $X$ such that $X \times X \cong Q$ but $X \not\cong Q$.

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**References**


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