ON \(K\)-SEMIMETRIC SPACES

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Abstract. An example is constructed of a separable Moore space that does not possess a compatible \(K\)-semimetric.

A semimetric \(d\) for a space \(X\) is said to be a \(K\)-semimetric if \(d(H, K) > 0\) whenever \(H\) and \(K\) are disjoint compact subsets of \(X\). It is the purpose of this note to provide an example of a regular semimetrizable space (in fact, a separable Moore space) which does not have a compatible \(K\)-semimetric. This answers a question first posed by A. V. Arhangel'skii in [1] and later by others such as H. Martin in [3].

The description of the example follows below. The sets \(R\), \(P\), \(Q\), and \(N\) denote the real numbers, irrational numbers, rational numbers, and natural numbers respectively.

Example 1. A separable Moore space which is not \(K\)-semimetrizable.

Let \(A_1 = P \times \{0\}\), \(A_2 = P \times \{-1\}\), \(E = \{(r, s) \in Q \times Q : s > 0\}\), and \(X = A_1 \cup A_2 \cup E\). Describe a local base for points in \(X\) as follows: Points in \(E\) have the usual neighborhoods (as inherited from \(Q \times R\)). If \(a \in P\) and \(n \in N\) let

\[
U_n(a, 0) = \{(a, 0)\} \cup \{(r, s) \in E : a < r < s/n + a, s < 1/n\},
\]

\[
U_n(a, -1) = \{(a, -1)\} \cup \{(r, s) \in E : -s/n + a < r < a, s < 1/n\}.
\]

Then \(\{U_n(a, 0)\}_n\) and \(\{U_n(a, -1)\}_n\) give local bases at \((a, 0) \in A_1\) and \((a, -1) \in A_2\) respectively. (A simple sketch reveals that \(U_n(a, 0)\) is \((a, 0)\) along with the “right half of the interior of a \(V\) neighborhood at \((a, 0)\)” and \(U_n(a, -1)\) is \((a, -1)\) along with the “left half”.) It is easily verified that \(X\) (with the new topology) is a separable, completely regular Moore space.

Let \(d\) be a semimetric for \(X\)–we show that \(d\) is not a \(K\)-semimetric.

For \(n \in N\) let

\[
P(n) = \{a \in P : d(x, y) > 1/n, \text{ all } x \in U_n(a, 0), y \in U_n(a, -1)\}.
\]

If \(P = \bigcup_{n=1}^{\infty} P(n)\) there is some \(k \in N\) such that \(T = \text{int}_R(\text{cl}_R, P(k)) \neq \emptyset\). It is possible to find \((t_1, s), (t_2, s) \in E \cap (T \times R)\) and \(b \in P(k)\) such that

\[
d((t_1, s), (t_2, s)) < \frac{1}{k}, \quad |t_1 - t_2| < \frac{1}{k(k + 1)}, \quad s = \frac{1}{k + 1}
\]

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and \( t_1 < b < t_2 \). If \( x = (t_2, s), y = (t_1, s) \) then \( x \in U_k(b, 0), y \in U_k(b, -1) \) and \( d(x, y) < 1/k \) which contradicts the definition of \( P(k) \). It follows that 

\[
a \notin P \left( \bigcup_{n=1}^{\infty} P(n) \right).
\]

Now \( a \notin P(n) \) implies there exists \( x_n \in U_n(a, 0) \) and \( y_n \in U_n(a, -1) \) such that \( d(x_n, y_n) < 1/n \). Clearly \( x_n \to (a, 0) \) and \( y_n \to (a, -1) \). If \( H = \{x_n\}^\infty_1 \cup \{(a, 0)\} \) and \( K = \{y_n\}^\infty_1 \cup \{(a, -1)\} \) then \( K \) and \( H \) are disjoint compact subsets of \( X \), but \( d(H, K) = 0 \); thus \( d \) is not a \( K \)-semimetric for \( X \) and the proof is complete.

**Remarks 2.** (a) It is trivial that every metric space is \( K \)-semimetrizable. In fact, A. V. Arhangel’skii [1] has shown that a regular space \( Y \) is metrizable if and only if there is a compatible semimetric \( d \) for \( Y \) such that \( d(A, B) > 0 \) whenever \( A \) and \( B \) are disjoint subsets of \( Y \) with \( A \) compact and \( B \) closed.

(b) It is known [1] that a semimetric space which is submetrizable (has a weaker metric topology) is also \( K \)-semimetrizable. W. Lindgren has pointed out to the author that a semimetric space with a coarser \( T_2 \) quasimetric topology is \( K \)-semimetrizable and that apparently, Example 1 gives the first known example of a Moore space that does not admit a coarser \( T_2 \) quasimetric topology.

(c) Let \( X \) be the space of Example 1 and let \( Y \) be the quotient space obtained from \( X \) by identifying points \((a, 0)\) and \((a, -1)\) for each \( a \in P \). Then \( Y \) is a completely regular separable submetrizable Moore space. If \( f: X \to Y \) is the corresponding quotient mapping then \( f \) is a perfect map from the nonsubmetrizable space \( X \) onto a submetrizable Moore space \( Y \). This should be contrasted with the result by Borges [2] and Okuyama [4] that if \( g: Z \to M \) is a perfect map from a Hausdorff space \( Z \) onto a metric space \( M \) then \( Z \) is metrizable if and only if \( Z \) has a \( G_\delta \)-diagonal. This suggests the following question.

**Question 1.** If \( g: Z \to M \) is a perfect map from a regular space \( Z \) onto a submetrizable space \( M \) what minimal diagonal condition on \( Z \) will ensure that \( Z \) is submetrizable? G. M. Reed has an example [6, Example 3] that shows \( Z \) need not be submetrizable even if \( Z \) has a regular \( G_\delta \)-diagonal.

Besides the submetrizable condition mentioned in Remark 2(b) different authors have given various sufficient conditions for a Moore space to be \( K \)-semimetrizable. H. Martin showed that a locally connected rim compact space is \( K \)-semimetrizable if and only if it is a developable \( \gamma \)-space [3]. A result by P. Zenor shows that a Moore space with a regular \( G_\delta \)-diagonal is \( K \)-semimetrizable [5].

**Question 2.** What minimal topological condition on a Moore space (or semimetric space) will ensure that the space be \( K \)-semimetrizable? For
example, is every locally connected rim compact Moore space necessarily $K$-semimetrizable?

**References**


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