A NEW AND CONSTRUCTIVE PROOF
OF THE BORSUK-ULAM THEOREM

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Abstract. The Borsuk-Ulam Theorem [1] states that if \( f \) is a continuous function from the \( n \)-sphere to \( n \)-space (\( f: S^n \to \mathbb{R}^n \)) then the equation \( f(x) = f(-x) \) has a solution. It is usually proved by contradiction using rather advanced techniques. We give a new proof which uses only elementary techniques and which finds a solution to the equation. If \( f \) is piecewise linear our proof is constructive in every sense; it is even easily implemented on a computer.

The Borsuk-Ulam Theorem has applications to fixed-point theory and corollaries include the Ham Sandwich Theorem and Invariance of Domain. The method used here is similar to Eaves [2] and Eaves and Scarf [3].

We use the following notational conventions. Let \( S^n = \{ x = (x_0, \ldots, x_n) \in \mathbb{R}^{n+1} | \text{some } x_i = \pm 1 \} \), the boundary of a cube. Note that the antipodal map \( \alpha: S^n \to S^n \), defined by \( \alpha(x) = -x \), is a PL homeomorphism. We use \( s, t \in \mathbb{R}; p, p', z \in \mathbb{R}^n; x, y \in S^n \subset \mathbb{R}^{n+1}; v \in S^n \times I \subset \mathbb{R}^{n+2}; \) and by tuples such as \( (p, s) \) or \( (z, s, t) \) we mean the obvious points of \( \mathbb{R}^{n+1} \) or \( \mathbb{R}^{n+2} \). A singleton set, such as \{\( \} \), will be represented without brackets. The origin in \( \mathbb{R}, \mathbb{R}^n, \) and \( \mathbb{R}^{n+1} \) will be represented by \( 0 \). We will let \( G_t(x) = G(x, t) \).

The Piecewise Linear Borsuk-Ulam Theorem. Let \( f: S^n \to \mathbb{R}^n \) be a PL map. Then there exists an \( x \in S^n \) such that \( f(x) = f(-x) \).

Proof. Since \( f \) is PL, it is linear on each simplex of a triangulation \( T \) of \( S^n \). Let \( T \cap \alpha T \) denote the subdivision of \( T \) into convex cells obtained by intersecting each simplex of \( T \) with the image of a simplex of \( T \) under \( \alpha \). Then \( f \) is linear on each cell of \( T \cap \alpha T \) and \( T \cap \alpha T \) is invariant under \( \alpha \). We can subdivide \( S^n \times I \) into convex cells by crossing each cell of \( T \cap \alpha T \) with \( I \).

We next subdivide these convex cells without adding new vertices to get a triangulation \( T^* \) which is still invariant under the homeomorphism

\[
H = \alpha \times \text{id}: S^n \times I \to S^n \times I
\]

\((H(x, t) = (-x, t)).\) To do this, order the pairs of vertices \((v, H(v))\). Note

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that $v$ and $H(v)$ cannot lie in a single convex cell of the subdivision. Suppose we have subdivided the $(r - 1)$-skeleton. Let $C$ be an $r$-cell and let $v$ be a vertex of $C$ from the first vertex pair meeting $C$. Use $v$ to cone on the faces of $C$ not containing $v$. (See [5, Problem 2.9] for the details of this argument.)

Choose $p \in \mathbb{R}^n$ so that $(p, 1, 1)$ lies in the interior of an $n$-simplex of $T^*$. Let $G_0(x) = f(x) - f(-x)$, for all $x \in S^n$, and let $G_1(z, s) = z - sp$ for all $(z, s) \in S^n$. Extend piecewise linearly to $G: S^n \times I \to \mathbb{R}^n$ using $T^*$. Note that $G(x, t) = -G(-x, t)$ and that $G_1^{-1}(0) = \{(p, 1), (-p, -1)\}$.

If $G((\sigma - S^n) \times 0)$ contains 0 for any $(n - 1)$-simplex of $T^*$, we make the following adjustments in $G$ (otherwise take $p' = p$). Make no change in $G|S^n \times 0$. Adjust the values of $G$ simultaneously on each pair $(z, s, 1)$ and $(-z, -s, 1)$ of vertices of $T^*$, redefining $G$ by extending piecewise linearly using $T^*$, so that:

(a) $G(z, s, 1) = -G(-z, -s, 1)$, for all $(z, s, 1) \in S^n \times 1$,
(b) For some $p' \in \mathbb{R}^n$, $G_1^{-1}(0) = \{(p', 1), (-p', -1)\}$, and
(c) No $G((\sigma - S^n) \times 0)$ contains 0, for any $\sigma$ in $T^*$ of dimension at most $n - 1$.

(b) is achieved by making the change in $G$ small. For (c), suppose that we are adjusting at $v$ and $\sigma^*v$ contradicts (c) while $\sigma$ satisfies (c). Then any adjustment of $G(v)$ out of the plane determined by $G(\sigma^*v)$ will make $\sigma^*v$ satisfy (c).

Now let $A$ be the component of $G^{-1}(0) - (S^n \times 0)$ containing $(p', 1, 1)$. $A$ is a polygonal arc which has its other endpoint either in $S^n \times 1$ or $S^n \times 0$ (in the latter case $A$ does not contain this endpoint). Then since $G(x, t) = -G(-x, t)$, $H(A)$ will be the component of $G^{-1}(0) - (S^n \times 0)$ containing $(-p', -1, 1)$. Either $A = H(A)$ or $A \cap H(A) = \phi$. The latter case holds since otherwise $A$ is a closed arc and $H$ would have to have a fixed point (by a PL version of the Intermediate Value Theorem). Hence $cl(A)$ must be an arc connecting $S^n \times 1$ to $S^n \times 0$. So $cl(A) \cap (S^n \times 0)$ is a solution.

Thus the algorithm for finding a solution consists of following a polygonal arc in $G^{-1}(0)$ from $S^n \times 1$ to $S^n \times 0$. This algorithm can be implemented numerically using techniques similar to those used to implement the simplex method of linear programming. See [2] for details. In practice, the adjustment of $G$ could be done in the process of following the arc. When the arc is found to intersect a simplex of dimension less than $n$, then $G$ could be adjusted to remove the intersection.

**Corollary (The Borsuk-Ulam Theorem).** Let $f: S^n \to \mathbb{R}^n$ be any continuous map. Then there exists an $x \in S^n$ so that $f(x) = f(-x)$.

**Proof.** Define $f^k: S^n \to \mathbb{R}^n$ by taking a triangulation of $S^n$ of mesh less than $1/k$, setting $f^k(x) = f(x)$ at the vertices of the triangulation and extending linearly. Then $f^k \to f$ uniformly, and there exists $x_k \in S^n$ so that $f^k(x_k) = f^k(-x_k)$. It follows that a subsequence of $\{x_k\}$ converges to some $x$ and $f(x) = f(-x)$. \qed
One cannot hope to generalize this result much by changing the antipodal map. For Pannwitz [4] gives an example in which \( \gamma: \mathbb{S}^n \to \mathbb{S}^n \) is a homeomorphism isotopic to the antipodal map which takes antipodal points to antipodal points and there is no solution to \( f(x) = f(\gamma(x)) \). In fact, by changing \( -\beta|\beta| \) to \( -\beta|\beta|^\epsilon, \epsilon > 0 \) at the bottom of p. 184 of [4], \( \gamma \) still has the above properties and can also be made arbitrary close to the antipodal map.

**Added in proof.** J. C. Alexander and J. A. Yorke have independently found a constructive proof of the Borsuk-Ulam Theorem. Their result is contained in the paper *The homotopy continuation method: numerically implementable topological procedures*, Trans. Amer. Math. Soc. (to appear).

**References**