A GENERAL RAMSEY PRODUCT THEOREM

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Abstract. Call a family $\mathcal{F}$ of subsets of a set $U$ *Ramsey* if no partition of $U$ into finitely many parts can split every $F \in \mathcal{F}$. We show that under very general conditions an arbitrary collection of Ramsey families in fact has a much stronger uniform Ramsey property.

A family $\mathcal{F}$ of finite subsets of a set $U$ is said to be *Ramsey* if for all integers $r < \infty$ and all mappings $\chi: U \to \{1, 2, \ldots, r\} \equiv [1, r]$, there exists an $F \in \mathcal{F}$ which is *homogeneous*, i.e., such that for some $i \in [1, r]$ $F \subseteq \chi^{-1}(i)$. Given an arbitrary mapping $P: U \times U \to U$, a family $\mathcal{F}$ is said to be a $P$-ideal of $U$ if

$$F \in \mathcal{F} \Rightarrow P(F, u) \in \mathcal{F}, \quad P(u, F) \in \mathcal{F},$$

for all $u \in U$, where $P$ is extended to $2^U \times 2^U$ in the usual way, i.e., for $X, Y \subseteq U$,

$$P(X, Y) \equiv \{ P(x, y): x \in X, y \in Y \}.$$

The following somewhat unexpected result shows that the Ramsey property holds simultaneously for arbitrary collections of Ramsey families under quite general conditions.

**Theorem.** Let $\{\mathcal{F}_a\}_{a \in A}$ be an arbitrary family of Ramsey $P$-ideals of $U$ where $P: U \times U \to U$ is arbitrary. Then for any $r < \infty$ and any mapping $\chi: U \to [1, r]$, there exists $i \in [1, r]$ and $F_a \in \mathcal{F}_a$, $a \in A$, such that $F_a \subseteq \chi^{-1}(i)$ for all $a \in A$.

**Proof.** We first show by induction that for any integer $m$ and any finite subcollection $\mathcal{F}_{a_1}, \ldots, \mathcal{F}_{a_t}$ of $\{\mathcal{F}_a\}_{a \in A}$, there is a finite set $F = [\mathcal{F}_{a_1}, \ldots, \mathcal{F}_{a_t}] \subseteq U$ such that for any mapping $\chi: F \to [1, m]$, there is an $i \in [1, m]$ and $F_j \in \mathcal{F}_{a_j}$ such that $F_j \subseteq \chi^{-1}(i)$ for $1 \leq j \leq t$. For $t = 1$, this follows at once from a well-known compactness principle (see [1]). Let $i > 1$ be fixed and suppose the assertion holds for all $t < i$. Also, the assertion is immediate for $m = 1$. Thus, let $m > 1$ be fixed and suppose the assertion also holds for $t = i$ and all $m < m$. Let $\mathcal{F}_{a_1}, \ldots, \mathcal{F}_{a_t}$ be an arbitrary fixed subcollection of $\{\mathcal{F}_a\}_{a \in A}$. By induction, the sets

$$X = [\mathcal{F}_{a_1}, \ldots, \mathcal{F}_{a_{i-1}}]_{m^*}, \quad Y = [\mathcal{F}_{a_i}]_{m} \quad \text{where } m^* = \overline{m}^{|X|},$$
and

\[ F^* = P(X, Y) \]

exist and are finite.

Let \( \chi: U \to [1, \bar{m}] \) be an arbitrary fixed mapping of \( U \) into \([1, \bar{m}]\). Define a new mapping \( \chi^* \) on \( Y \) so that

\[ \chi^*(y) = \chi^*(y'), \quad y, y' \in Y, \]

iff

\[ \chi(P(x, y)) = \chi(P(x, y')) \quad \text{for all } x \in X. \]

Since

\[ |P(X, y)| < |X| \quad \text{for all } y \in Y \]

then we can take \( \chi^* \) to be a mapping of \( Y \) into \([1, m^*]\). By the definition of \( Y \), there exists \( F_i \in \mathcal{F}_a \) such that for some \( i \in [1, m^*] \), \( F_i \subseteq \chi^{*^{-1}}(i) \). Let \( f \in F_i \).

We now define another mapping \( \chi': X \to [1, \bar{m}] \) by letting

\[ \chi'(x) = \chi(P(x, f)), \quad x \in X. \]

Note that the value of \( \chi' \) is actually independent of the choice of \( f \).

By the definition of \( X \), there exists \( k \in [1, \bar{m}] \) and \( F_j \in \mathcal{F}_a \) such that

\[ F_j \subseteq \chi^{-1}(k), \quad 1 < j < \bar{i} - 1. \]

Therefore,

\[ P(F_j, f) \subseteq \chi^{-1}(k), \quad 1 < j < \bar{i} - 1, \]

and so

\[ P(F_i, F_j) \subseteq \chi^{-1}(k), \quad 1 < j < \bar{i} - 1, \]

since

\[ \chi(P(x, f)) = \chi(P(x, f')), \quad x \in X, \quad f, f' \in F_i. \]

But

\[ P(F_j, f) \subseteq \mathcal{F}_a, \quad 1 < j < \bar{i} - 1, \]

since \( F_a \) is a \( P \)-ideal, and \( P(x, F_i) \subseteq \mathcal{F}_a \) for the same reason. Since all \( t \) of these sets are in \( \chi^{-1}(k) \) then we have shown that \( P(X, Y) \) can be taken as \([\mathcal{F}_a, \ldots, \mathcal{F}_a]_{\bar{m}} \). This completes the induction step and the first assertion is proved.

Now, suppose the theorem fails. Thus, for some \( r \) there is a mapping \( \chi: U \to [1, r] \) and families \( \mathcal{F}_i \in \{ \mathcal{F}_a \}_{a \in A} \) such that

\[ F_i \subseteq \chi^{-1}(i) \quad \text{for all } F_i \in \mathcal{F}_i, \quad 1 < i < r. \quad (1) \]

By the preceding assertion, the (finite) set

\[ [\mathcal{F}_1, \ldots, \mathcal{F}_r] \subseteq U \]

exists. Thus, for some \( k \in [1, r] \) and \( F'_j \in \mathcal{F}_j \),

\[ F'_j \subseteq \chi^{-1}(k), \quad 1 < j < r. \]

In particular, \( F'_k \subseteq \chi^{-1}(k) \) and \( F'_k \in \mathcal{F}_a \). This contradicts (1) and the theorem is proved. \[ \square \]
Typical applications of this theorem can produce significant strengthenings of many of the standard Ramsey-type results. For example, an old result of Gallai (see [3]), generalizing the theorem of van der Waerden on arithmetic progressions (see [2], [4]), asserts that for any finite subset $C$ of $E^n$, in any partition of $E^n$ into finitely many classes, some class always contains a subset $C'$ which is similar to $C$. Using the product theorem of this note, taking $U$ to be $E^n$ and for $\bar{x}, \bar{y} \in E^n$, defining $P(\bar{x}, \bar{y}) = \bar{x} + \bar{y}$, we see, in fact, that in any partition of $E^n$ into finitely many classes, one class must contain similar copies of every finite subset of $E^n$.

By taking $U = Z^+$, the set of positive integers, and $P(x, y) = xy$, we obtain the following classical theorem of Rado [3]. Call a system $\mathcal{S}$ of homogeneous, linear equations regular, if for any partition of $Z^+$ into finitely many classes, $\mathcal{S}$ has a solution entirely in one class. (Such systems were completely characterized by Rado.) Then, in fact, by the product theorem, for any partition of $Z^+$ into finitely many classes, some class contains solutions to every regular system of equations.

References


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