RECURSIVE DETERMINATION OF THE SUM-OF-DIVISORS FUNCTION

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Abstract. A recursive scheme for determination of the sum-of-divisors function is presented. As all of the formulas involve triangular numbers, the scheme is therefore compared for efficiency with another known recursive triangular-number formula for this function.

For each positive integer \( n \), \( \sigma(n) \) denotes the sum of the positive divisors of \( n \). Recently Ewell [1] derived two recursive schemes for computing the values \( \sigma(n) \), and regarding efficiency of computation compared each of them with the well-known pentagonal-number formula of Euler. In the present discussion we derive another recursive scheme and show that among the four known recursive determinations of \( \sigma \) this one ranks no worse than second with regard to efficiency. We prepare the way for an easy statement of our result by defining an auxiliary function \( \alpha \) as follows. For each positive integer \( n \), we express \( n \) uniquely as \( n = 2^{b(n)} O(n) \) where \( b(n) \) is a nonnegative integer and \( O(n) \) is odd. Now define \( \alpha(n) = (2^{b(n)} + 1 - 3)\sigma(O(n)) \).

Theorem. For each positive integer \( m \),

\[
\sigma(2m - 1) = \sum_{k=0} \left\{ \alpha(2m - 1 - (2k + 1)(4k + 1)) + \alpha(2m - 1 - (2k + 1)(4k + 3)) \right\} + \sum_{k=0} \left\{ \sigma(2m - 1 - (2k + 2)(4k + 3)) + \sigma(2m - 1 - (2k + 2)(4k + 5)) \right\}
\]

\[= \begin{cases} 
  n(n + 1)/2, & \text{if } 2m - 1 = n(n + 1)/2, \\
  0, & \text{otherwise},
\end{cases} \quad (1)
\]
\[
\alpha(2m) - \sum_{k=0} \{ \sigma(2m - (2k + 1)(4k + 1)) + \sigma(2m - (2k + 1)(4k + 3)) \} \\
+ \sum_{k=0} \{ \alpha(2m - (2k + 2)(4k + 3)) + \alpha(2m - (2k + 2)(4k + 5)) \} \\
= \begin{cases} 
-n(n+1)/2, & \text{if } 2m = n(n+1)/2, \\
0, & \text{otherwise},
\end{cases} 
\] (2)

where in both (1) and (2) summation extends as far as the arguments are positive.

PROOF. Our proof depends on the following two identities due respectively to Euler and Gauss.

\[
\prod_{n=1}^{\infty} (1 + x^n)(1 - x^{2n-1}) = 1,
\]

\[
\prod_{n=1}^{\infty} (1 - x^{2n})(1 + x^n) = \sum_{n=0}^{\infty} x^{n(n+1)/2}.
\]

For proofs see [2, pp. 277–284]. Since \(1 - x^{2n} = (1 - x^n)(1 + x^n)\), we use Euler’s identity to express Gauss’s identity in the equivalent form

\[
\prod_{n=1}^{\infty} (1 - x^n) \prod_{n=1}^{\infty} (1 - x^{2n-1})^{-2} = \sum_{n=0}^{\infty} x^{n(n+1)/2}.
\]

Briefly, set \(F(x) = \sum x^{n(n+1)/2}\). Now take the logarithmic derivative of the foregoing identity and multiply the resulting identity by \(x\) to obtain

\[
\sum_{n=1}^{\infty} \frac{nx^n}{1 - x^n} - 2 \sum_{n=1}^{\infty} \frac{(2n-1)x^{2n-1}}{1 - x^{2n-1}} = -x F'(x)/F(x). \tag{3}
\]

It is well known that the first series on the left of (3), a “Lambert” series, generates \(\sigma(n)\): i.e.,

\[
\sum_{n=1}^{\infty} \frac{nx^n}{1 - x^n} = \sum_{n=1}^{\infty} \sigma(n)x^n.
\]

The second series is perhaps less well known, but straightforward algebraic manipulation shows that

\[
\sum_{n=1}^{\infty} \frac{(2n-1)x^{2n-1}}{1 - x^{2n-1}} = \sum_{n=1}^{\infty} x^n \sum_{d|n, d \text{ odd}} d
\]

\[
= \sum_{m=1}^{\infty} \sigma(2m-1)x^{2m-1} + \sum_{m=1}^{\infty} \sigma(O(2m))x^{2m}.
\]

Thus, identity (3) becomes

\[
\sum_{m=1}^{\infty} \sigma(2m-1)x^{2m-1} - \sum_{m=1}^{\infty} \alpha(2m)x^{2m} = xF'(x)/F(x).
\]
Now, separating the even and odd triangular numbers $n(n + 1)/2$ by the least positive residues of $n \pmod{4}$, we write

\[
\left\{ \sum_{m=1}^{\infty} \sigma(2m - 1)x^{2m-1} \right\} F(x)
= \sum_{m=1}^{\infty} x^{2m-1} \left\{ \sigma(2m - 1) + \sum_{k=0}^{\infty} \left[ \sigma(2m - 1 - (2k + 2)(4k + 3)) + \sigma(2m - (2k + 1)(4k + 1)) \right] \right. \\
\left. + \sigma(2m - (2k + 1)(4k + 3)) \right\}
+ \sum_{m=1}^{\infty} x^{2m} \left\{ \sum_{k=0}^{\infty} \left[ \sigma(2m - (2k + 1)(4k + 1)) + \sigma(2m - (2k + 1)(4k + 3)) \right] \right. \\
\left. + \sigma(2m - (2k + 2)(4k + 3)) \right\},
\]

and,

\[
\left\{ \sum_{m=1}^{\infty} \alpha(2m)x^{2m} \right\} F(x)
= \sum_{m=1}^{\infty} x^{2m-1} \left\{ \sum_{k=0}^{\infty} \left[ \alpha(2m - 1 - (2k + 1)(4k + 1)) + \alpha(2m - 1 - (2k + 1)(4k + 3)) \right] \right. \\
\left. + \alpha(2m - (2k + 1)(4k + 3)) \right\}
+ \sum_{m=1}^{\infty} x^{2m} \left\{ \alpha(2m) + \sum_{k=0}^{\infty} \left[ \alpha(2m - (2k + 2)(4k + 3)) + \alpha(2m - (2k + 2)(4k + 5)) \right] \right. \\
\left. + \alpha(2m - (2k + 2)(4k + 5)) \right\}. 
\]

We then substitute these last two developments into the identity

\[
\left\{ \sum_{m=1}^{\infty} \sigma(2m - 1)x^{2m-1} \right\} F(x) - \left\{ \sum_{m=1}^{\infty} \alpha(2m)x^{2m} \right\} F(x)
= xF'(x) = \sum_{n=1}^{\infty} \frac{n(n + 1)}{2} x^{n(n+1)/2},
\]
equate coefficients of odd powers $x^{2m-1}$ to obtain recurrence (1) and equate coefficients of even powers $x^{2m}$ to obtain recurrence (2).

**Remarks.** One of the three recurrences discussed in [1] is

\[
\sum_{k=0}^{\infty} (-1)^{k} (2k + 1) \sigma(n - k(k + 1)/2)
= \begin{cases} 
(\alpha)^{j+1} j(j + 1)(2j + 1)/6, & \text{if } n = j(j + 1)/2, \\
0, & \text{otherwise}, 
\end{cases}
\]

(4)
Like our recurrences (1) and (2) this recurrence involves triangular numbers. Since \( \sigma \) is multiplicative and therefore \( \sigma(n) = (2^{k(n)+1} - 1)\sigma(\Omega(n)) \), we suppose that we are given a large odd number \( n \) and investigate efficiency of computation of \( \sigma(n) \) by recurrences (1) and (4). Theoretically we deduce that each of the recurrences needs about \( \sqrt{2n} \) of the values, \( \sigma(j), 1 < j < n \). But, practically, let us take a not-too-large value of \( n \), say \( n = 63 \), partially compute \( \sigma(63) \) by each of the recurrences and possibly observe some noteworthy differences.

By recurrence (1),

\[
\sigma(63) = (2^2 - 3)\sigma(31) + (2^5 - 3)\sigma(3) + (2^2 - 3)\sigma(9) + (2^3 - 3)\sigma(15) \\
+ (2^2 - 3)\sigma(21) + (2^4 - 3)\sigma(1) - \sigma(57) - \sigma(35) - \sigma(53) - \sigma(27).
\]

By recurrence (4),

\[
\sigma(63) = 3\sigma(62) - 5\sigma(60) + 7\sigma(57) - 9\sigma(53) + 11\sigma(48) - 13\sigma(42) \\
+ 15\sigma(35) - 17\sigma(27) + 19\sigma(18) - 21\sigma(8).
\]

Although each recurrence uses 10 lower values, we observe that recurrence (1) uses smaller values \( \sigma(j) \) by separating the binary and odd parts of \( j \). (This is something that any high-speed computing machine can do easily.) Also, recurrence (1) avoids coefficients such as the \( 2k + 1 \) of (4).

**References**


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