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RECURSIVE DETERMINATION OF THE SUM-OF-DIVISORS FUNCTION

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ABSTRACT. A recursive scheme for determination of the sum-of-divisors function is presented. As all of the formulas involve triangular numbers, the scheme is therefore compared for efficiency with another known recursive triangular-number formula for this function.

For each positive integer n, $\sigma(n)$ denotes the sum of the positive divisors of n. Recently Ewell [1] derived two recursive schemes for computing the values $\sigma(n)$, and regarding efficiency of computation compared each of them with the well-known pentagonal-number formula of Euler. In the present discussion we derive another recursive scheme and show that among the four known recursive determinations of σ this one ranks no worse than second with regard to efficiency. We prepare the way for an easy statement of our result by defining an auxiliary function α as follows. For each positive integer n, we express n uniquely as $n = 2^{b(n)}O(n)$ where $b(n)$ is a nonnegative integer and $O(n)$ is odd. Now define $\alpha(n) = (2^{b(n)+1} - 3)\sigma(O(n))$.

THEOREM. *For each positive integer* m,

$$\begin{aligned}\sigma(2m-1) - \sum_{k=0} \{\alpha(2m-1-(2k+1)(4k+1)) \\ + \alpha(2m-1-(2k+1)(4k+3))\} \\ + \sum_{k=0} \{\sigma(2m-1-(2k+2)(4k+3)) \\ + \sigma(2m-1-(2k+2)(4k+5))\}\end{aligned}$$

$$= \begin{cases} n(n+1)/2, & \text{if } 2m-1 = n(n+1)/2, \\ 0, & \text{otherwise}, \end{cases} \tag{1}$$

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and

$$\alpha(2m) - \sum_{k=0} \{\sigma(2m-(2k+1)(4k+1)) + \sigma(2m-(2k+1)(4k+3))\}$$
$$+ \sum_{k=0} \{\alpha(2m-(2k+2)(4k+3)) + \alpha(2m-(2k+2)(4k+5))\}$$
$$= \begin{cases} -n(n+1)/2, & \text{if } 2m = n(n+1)/2, \\ 0, & \text{otherwise,} \end{cases} \tag{2}$$

where in both (1) *and* (2) *summation extends as far as the arguments are positive.*

PROOF. Our proof depends on the following two identities due respectively to Euler and Gauss.

$$\prod_{n=1}^{\infty} (1+x^n)(1-x^{2n-1}) = 1,$$

$$\prod_{n=1}^{\infty} (1-x^{2n})(1+x^n) = \sum_{n=0}^{\infty} x^{n(n+1)/2}.$$

For proofs see [2, pp. 277–284]. Since $1 - x^{2n} = (1-x^n)(1+x^n)$, we use Euler's identity to express Gauss's identity in the equivalent form

$$\prod_{n=1}^{\infty} (1-x^n) \prod_{n=1}^{\infty} (1-x^{2n-1})^{-2} = \sum_{n=0}^{\infty} x^{n(n+1)/2}.$$

Briefly, set $F(x) = \sum x^{n(n+1)/2}$. Now take the logarithmic derivative of the foregoing identity and multiply the resulting identity by x to obtain

$$\sum_{n=1}^{\infty} \frac{nx^n}{1-x^n} - 2\sum_{n=1}^{\infty} \frac{(2n-1)x^{2n-1}}{1-x^{2n-1}} = -\frac{xF'(x)}{F(x)}. \tag{3}$$

It is well known that the first series on the left of (3), a "Lambert" series, generates $\sigma(n)$: i.e.,

$$\sum_{n=1}^{\infty} \frac{nx^n}{1-x^n} = \sum_{n=1}^{\infty} \sigma(n)x^n.$$

The second series is perhaps less well known, but straightforward algebraic manipulation shows that

$$\sum_{n=1}^{\infty} \frac{(2n-1)x^{2n-1}}{1-x^{2n-1}} = \sum_{n=1}^{\infty} x^n \sum_{\substack{d|n \\ d \text{ odd}}} d$$
$$= \sum_{m=1}^{\infty} \sigma(2m-1)x^{2m-1} + \sum_{m=1}^{\infty} \sigma(O(2m))x^{2m}.$$

Thus, identity (3) becomes

$$\sum_{m=1}^{\infty} \sigma(2m-1)x^{2m-1} - \sum_{m=1}^{\infty} \alpha(2m)x^{2m} = xF'(x)/F(x).$$

Now, separating the even and odd triangular numbers $n(n+1)/2$ by the least positive residues of n (mod 4), we write

$$\left\{\sum_{m=1}^{\infty} \sigma(2m-1)x^{2m-1}\right\}F(x)$$
$$= \sum_{m=1}^{\infty} x^{2m-1}\left\{\sigma(2m-1) + \sum_{k=0}\left[\sigma(2m-1-(2k+2)(4k+3))\right.\right.$$
$$\left.\left.+\ \sigma(2m-1-(2k+2)(4k+5))\right]\right\}$$
$$+ \sum_{m=1}^{\infty} x^{2m}\left\{\sum_{k=0}\left[\sigma(2m-(2k+1)(4k+1))\right.\right.$$
$$\left.\left.+\ \sigma(2m-(2k+1)(4k+3))\right]\right\},$$

and,

$$\left\{\sum_{m=1}^{\infty} \alpha(2m)x^{2m}\right\}F(x)$$
$$= \sum_{m=1}^{\infty} x^{2m-1}\left\{\sum_{k=0}\left[\alpha(2m-1-(2k+1)(4k+1))\right.\right.$$
$$\left.\left.+\ \alpha(2m-1-(2k+1)(4k+3))\right]\right\}$$
$$+ \sum_{m=1}^{\infty} x^{2m}\left\{\alpha(2m) + \sum_{k=0}\left[\alpha(2m-(2k+2)(4k+3))\right.\right.$$
$$\left.\left.+\ \alpha(2m-(2k+2)(4k+5))\right]\right\}.$$

We then substitute these last two developments into the identity

$$\left\{\sum_{m=1}^{\infty} \sigma(2m-1)x^{2m-1}\right\}F(x) - \left\{\sum_{m=1}^{\infty} \alpha(2m)x^{2m}\right\}F(x)$$
$$= xF'(x) = \sum_{n=1}^{\infty} \frac{n(n+1)}{2}\, x^{n(n+1)/2},$$

equate coefficients of odd powers x^{2m-1} to obtain recurrence (1) and equate coefficients of even powers x^{2m} to obtain recurrence (2).

REMARKS. One of the three recurrences discussed in [1] is

$$\sum_{k=0} (-1)^k(2k+1)\sigma(n-k(k+1)/2)$$
$$= \begin{cases} (-1)^{j+1}j(j+1)(2j+1)/6, & \text{if } n = j(j+1)/2, \\ 0, & \text{otherwise.} \end{cases} \qquad (4)$$

Like our recurrences (1) and (2) this recurrence involves triangular numbers. Since σ is multiplicative and therefore $\sigma(n) = (2^{b(n)+1} - 1)\sigma(O(n))$, we suppose that we are given a large odd number n and investigate efficiency of computation of $\sigma(n)$ by recurrences (1) and (4). Theoretically we deduce that each of the recurrences needs about $\sqrt{2n}$ of the values, $\sigma(j)$, $1 \leq j < n$. But, practically, let us take a not-too-large value of n, say $n = 63$, partially compute $\sigma(63)$ by each of the recurrences and possibly observe some noteworthy differences.

By recurrence (1),

$$\sigma(63) = (2^2 - 3)\sigma(31) + (2^5 - 3)\sigma(3) + (2^2 - 3)\sigma(9) + (2^3 - 3)\sigma(15) + (2^2 - 3)\sigma(21) + (2^4 - 3)\sigma(1) - \sigma(57) - \sigma(35) - \sigma(53) - \sigma(27).$$

By recurrence (4),

$$\sigma(63) = 3\sigma(62) - 5\sigma(60) + 7\sigma(57) - 9\sigma(53) + 11\sigma(48) - 13\sigma(42) + 15\sigma(35) - 17\sigma(27) + 19\sigma(18) - 21\sigma(8).$$

Although each recurrence uses 10 lower values, we observe that recurrence (1) uses smaller values $\sigma(j)$ by separating the binary and odd parts of j. (This is something that any high-speed computing machine can do easily.) Also, recurrence (1) avoids coefficients such as the $2k + 1$ of (4).

References

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