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EXPANSIONS ASSOCIATED WITH NON-SELF-ADJOINT BOUNDARY-VALUE PROBLEMS

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ABSTRACT. Eigenfunction expansions are given for a class of non-self-adjoint boundary-value problems. The results apply to systems of ordinary differential equations, some of whose coefficients may be arbitrary bounded Hilbert space operators.

Introduction. Given a differential operator $\mathcal{L}$ on $\bigoplus^k L^2[0, 1]$ with compact resolvent, there are associated sequences of projections $\{P_m\}$ obtained by integrating the resolvent around circles whose radii tend to infinity. The problem of obtaining an eigenfunction expansion associated with $\mathcal{L}$ is just the problem of showing that for each $f \in \bigoplus^k L^2[0, 1]$, $\lim_{m\to\infty} \|f - P_m f\| = 0$. The usual technique for establishing such results involves an analysis of the kernel of the resolvent. If the operator is of high order, or one is dealing with a system of equations, this technique becomes unwieldy. We present an abstract approach to these problems which avoids the usual analysis of the resolvent kernel.

Denote the orthogonal sum $L^2[0, 1] \oplus \cdots \oplus L^2[0, 1]$ (k summands) by $\bigoplus^k L^2[0, 1]$. Let $D = d/dx$. A trivial but useful fact is that the operator $\mathcal{L} = iD$ acting componentwise on $\bigoplus^k L^2[0, 1]$ is self-adjoint if equipped with the domain of n-tuples $(f_1, \ldots, f_n)$ of absolutely continuous functions whose derivatives are in $L^2[0, 1]$, and such that $(f_1, \ldots, f_n)(0) = (f_1, \ldots, f_n)(1)$. Moreover the spectrum of $\mathcal{L}$ is $\{2\pi n | n \in Z\}$, and each eigenvalue has multiplicity k. We begin by showing that the distribution of eigenvalues for this example is typical for a large class of unbounded operators. Information about the eigenvalue distribution for higher order differential operators is easily obtained by use of the spectral mapping theorem.

Fixing notation, we let $\mathfrak{D}(\mathcal{L})$ be the domain of the operator $\mathcal{L}$, and $\sigma(\mathcal{L})$ its spectrum. Denote by C the complex numbers and by $\mathcal{L}^*$ the Hilbert space adjoint of the operator $\mathcal{L}$. Finally we remark that many of the basic facts about ordinary differential operators can be found in Goldberg [5].

I. Eigenvalue comparisons and the expansion result. Let H be a Hilbert space with two Hilbert space norms, $\|\cdot\|_1$ and $\|\cdot\|_2$ which induce the same topology on H. Assume that for all $f \in H$, $\|f\|_1 \leqslant \|f\|_2$. Let $\mathcal{L}$ and M be

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self-adjoint on H with norms $\|\cdot\|_1$ and $\|\cdot\|_2$ respectively. We suppose that $\mathfrak{L}$ and M have the same domain and that their spectra are discrete and do not include zero. A well-known application of the closed graph theorem shows that there is a positive constant β such that

$$\frac{1}{\beta}\|Mf\|_2 \leqslant \|\mathfrak{L}f\|_1, \qquad f \in \mathfrak{D}(\mathfrak{L}). \tag{†}$$

Define $\Pi_M((a, b))$ to be the number of eigenvalues of M in the interval (a, b), counted with multiplicity.

(1.1) LEMMA. $\Pi_{\mathfrak{L}}((-b, b)) \leqslant \Pi_M((-\beta b, \beta b))$.

PROOF. If not we can find ψ such that $\|\psi\|_1 = 1$, and ψ is in the span of the eigenfunctions of $\mathfrak{L}$ with eigenvalues in $(-b, b)$, while being orthogonal to the span of the eigenfunctions of M with eigenvalues in $(-\beta b, \beta b)$ with respect to the second inner product. Since $\|\psi\|_2 \geqslant 1$, we have $\|\mathfrak{L}\psi\|_1 < b$ while $\|M\psi\|_2 \geqslant \beta b$. This contradicts (†), so establishes the result. □

Suppose that $\mathfrak{L}$ and $\tilde{\mathfrak{L}}$ are self-adjoint nth order ordinary differential operators on $\bigoplus^k L^2[0, 1]$, and that if f is infinitely differentiable and compactly supported in $(0, 1)$, then $f \in \mathfrak{D}(\mathfrak{L}) \cap \mathfrak{D}(\tilde{\mathfrak{L}})$ and $\mathfrak{L}f = \tilde{\mathfrak{L}}f$. Moreover we suppose that $\mathfrak{L}f = \sum_{j=0}^n A_j(x)D^j f$ for all f as above, that A_j is j-times continuously differentiable on $[0, 1]$, and that $\det(A_n(x)) \neq 0$ for $x \in [0, 1]$.

(1.2) LEMMA. *For any open interval* $\mathcal{I}$, $|\Pi_{\mathfrak{L}}(\mathcal{I}) - \Pi_{\tilde{\mathfrak{L}}}(\mathcal{I})| \leqslant kn$.

PROOF. By adding a suitable real constant to $\mathfrak{L}$ and $\tilde{\mathfrak{L}}$ we can assume $\mathcal{I} = (-b, b)$, $b > 0$. Assume to the contrary that $\Pi_{\mathfrak{L}}((-b, b)) - \Pi_{\tilde{\mathfrak{L}}}((-b, b)) \geqslant kn + 1$. Since our differential operators fall into the regular case, there will be a function ψ of unit norm which is a linear combination of eigenfunctions of $\mathfrak{L}$ with eigenvalues in $(-b, b)$, $\psi \in \mathfrak{D}(\mathfrak{L}) \cap \mathfrak{D}(\tilde{\mathfrak{L}})$, and ψ is orthogonal to the eigenfunctions of $\tilde{\mathfrak{L}}$ with eigenvalues in $(-b, b)$. Thus we have $(\mathfrak{L}\psi, \mathfrak{L}\psi) = (\tilde{\mathfrak{L}}\psi, \tilde{\mathfrak{L}}\psi)$, but $(\mathfrak{L}\psi, \mathfrak{L}\psi) < b^2$ while $(\tilde{\mathfrak{L}}\psi, \tilde{\mathfrak{L}}\psi) \geqslant b^2$. This contradiction gives the result. □

Observe that if $\mathfrak{L}$ is formally self-adjoint, regular, and of order $2n$, then the boundary conditions

$$f(0) = \cdots = f^{(n-1)}(0) = f(1) = \cdots = f^{(n-1)}(1) = 0$$

will make $\mathfrak{L}$ a self-adjoint operator. Thus by using (1.1) and (1.2) and the remarks made earlier about iD on $\bigoplus^k L^2[0, 1]$, we see that for any regular self-adjoint ordinary differential operator $\mathfrak{L}$ of order n there is a positive constant c_1 such that $\Pi_{\mathfrak{L}}((-b, b)) \leqslant c_1 b^{1/n}$ if $b > 1$.

Consider the circles centered at the origin with radii m^n, $m = 1, 2, \ldots$. To preserve the bound of $c_1 b^{1/n}$ eigenvalues in $(-b, b)$, no more than half of the annuli bounded by the circles of radii $(m + 1)^n$ and m^n, for $m = 1, 2, \ldots, r$ can contain more than $2c_1$ eigenvalues of $\mathfrak{L}$. Consequently we have the following:

(1.3) LEMMA. *Let $\mathfrak{L}$ be a regular nth order self-adjoint ordinary differential operator on $\bigoplus^k L^2[0, 1]$. Then there is an infinite collection of circles S_m centered at zero with radii R_m such that $m^n < R_m < (m+1)^n$, and there is a positive constant c_2 such that the distance from S_m to $\sigma(\mathfrak{L})$ is at least $c_2 m^{n-1}$.*

Now let $\mathfrak{L}$ be as above and define a formal operator $\Phi = \sum_{i=0}^{n-2} A_i D^i$, where A_i is a bounded operator on $\bigoplus^k L^2[0, 1]$. Let $T = \mathfrak{L} - \Phi$, and $R(z)$, $\tilde{R}(z)$ denote $(\mathfrak{L} - zI)^{-1}$, $(T - zI)^{-1}$ respectively. The circles S_m are as in (1.3).

(1.4) THEOREM. *Except for a discrete set of $z \in C$, $\tilde{R}(z)$ exists. If $f \in \bigoplus^k L^2[0, 1]$ we have*

$$f = \lim_{m\to\infty} \frac{1}{2\pi i} \int_{S_m} \tilde{R}(z) f\, dz.$$

PROOF. Since $T - zI = [I - \Phi R(z)][\mathfrak{L} - zI]$ whenever $R(z)$ exists, we have

$$\tilde{R}(z) = R(z)\left[I - \Phi R(z)\right]^{-1} \quad \text{if } \|\Phi R(z)\| < 1.$$

Now

$$\|\Phi R(z)\| \leqslant (n-1) \sup_i \|A_i\| \sup_i \|D^i R(z)\|.$$

To estimate $\|D^i R(z)\|$ we use a lemma from Friedman, *Partial differential equations*, p. 19. There it is shown that

$$\|D^i u\|^2 \leqslant \varepsilon \|D^n u\|^2 + \frac{c}{\varepsilon^{i/(n-i)}} \|u\|^2 \quad \text{if } i < n, 0 < \varepsilon \leqslant \varepsilon_0, \qquad (\dagger\dagger)$$

where ε_0 and c depend only on n.

Since $\|R(z)\| = 1/d(z, \sigma(\mathfrak{L}))$, we have $\|R(z)\| \leqslant 1/c_2 m^{n-1}$ if $z \in S_m$. By $(\dagger\dagger)$, if we let $|z| = R_m$ and $\varepsilon = (1/|z|)^{4/n}$ we have $\|D^i R(z) f\| \leqslant (c_3/m)\|f\|$, $i \leqslant n-2$, where c_3 is a constant. Here we have used $\|D^n u\| \leqslant K(\|\mathfrak{L} u\| + \|u\|)$, K a constant. Thus for $z \in S_m$,

$$\|\Phi R(z)\| \leqslant (n-1) \sup_j \|A_j\| \frac{c_3}{m}. \qquad (*)$$

This establishes the existence of $\tilde{R}(z)$ for m large. The discreteness of $\sigma(T)$ follows immediately.

Now for $z \in S_m$ and $f \in \bigoplus^k L^2[0, 1]$,

$$\frac{1}{2\pi i} \int_{S_m} \tilde{R}(z) f\, dz = \frac{1}{2\pi i} \int_{S_m} R(z)\left[I - \Phi R(z)\right]^{-1} f\, dz$$

$$= \frac{1}{2\pi i} \int_{S_m} R(z) f\, dz + \frac{1}{2\pi i} \int_{S_m} R(z)\left[I - \Phi R(z)\right]^{-1} \Phi R(z) f\, dz.$$

Since $\mathfrak{L}$ is self-adjoint we know that $\lim_{m\to\infty}(1/2\pi i)\int_{S_m} R(z) f\, dz = f$. Thus we only need show that

$$\lim_{m\to\infty} \int_{S_m} R(z)\left[I - \Phi R(z)\right]^{-1} \Phi R(z) f\, dz = 0.$$

Let $\varepsilon > 0$ and $f = \sum_{i=1}^{\infty} b_i x_i$, where $\mathfrak{L} x_i = \lambda_i x_i$. Then we can find j such that $\|f - \sum_{i=1}^{j} b_i x_i\| < \varepsilon$. We have

$$\int_{S_m} R(z)\left[I - \Phi R(z)\right]^{-1} \Phi R(z) f\, dz$$

$$= \int_{S_m} R(z)\left[I - \Phi R(z)\right]^{-1} \Phi R(z) \left\{ \sum_{i=1}^{j} b_i x_i \right\} dz$$

$$+ \int_{S_m} R(z)\left[I - \Phi R(z)\right]^{-1} \Phi R(z) \left\{ f - \sum_{i=1}^{j} b_i x_i \right\} dz.$$

This last integral is bounded in norm by $R_m(1/c_2 m^{n-1})(c_4/m)\varepsilon$ by (*), where c_4 is a constant. To estimate the preceding integral we note that for $z \in S_m$ we have $\lim_{m\to\infty} \|R(z)\| = 0$, $\|[I - \Phi R(z)]^{-1}\|$ is bounded, and

$$\left\| \sum_{i=1}^{j} b_i \frac{\Phi x_i}{\lambda_i - z} \right\| = O\left(\frac{1}{|z|} \right).$$

Thus the preceding integral goes to zero in norm as $m \to \infty$. This completes the proof. □

The method of proof for this theorem is similar to one found in Clark [**1**]. He describes more detailed results when $\mathfrak{L} = (iD)^n$ on $L^2[0, 1]$ with certain separated boundary conditions.

II. An application. Once we have an eigenfunction expansion for a differential operator, we also have an expansion for every operator similar to the one we started with. In some cases it is not difficult to identify a large collection of operators similar to those whose expansions we described in (1.4). That is the aim of this section.

Consider an operator $\mathfrak{L}$ on $\bigoplus^k L^2[0, 1]$ formally given by $-D^2 + P$, where P is a bounded operator on $\bigoplus^k L^2[0, 1]$. To define the domain of $\mathfrak{L}$ we impose $2k$ separated boundary conditions on $f \in \bigoplus^k L^2[0, 1]$:

$$S_1 f(0) + S_2 f'(0) = 0, \quad T_1 f(1) + T_2 f'(1) = 0.$$

Here S_1, S_2, T_1, T_2 are $k \times k$ matrices such that the boundary functionals they define are linearly independent. Recall that for a $k \times k$ matrix B we define $\operatorname{Im} B = [B - B^*]/2i$.

(2.1) THEOREM. *If there are self-adjoint $k \times k$ matrices A_1, A_2 such that $-\frac{1}{2}\operatorname{Im}[S_2 S_1^*] = S_2 A_1 S_2^*$, and $-\frac{1}{2}\operatorname{Im}[T_2 T_1^*] = T_2 A_2 T_2^*$, then $\mathfrak{L}$ is unitarily equivalent to a bounded perturbation of a second order self-adjoint regular differential operator on $\bigoplus^k L^2[0, 1]$.*

The plan is to show the existence of a smooth $k \times k$ unitary matrix valued function $U(x)$ such that $U^*[-D^2 + P]U = [iD + Q]^2 + C$, where Q is a bounded self-adjoint $k \times k$ matrix-valued function and C is a bounded operator. The differential expression $[iD + Q]^2$ is formally self-adjoint, and this new operator has defining boundary conditions

$$S_1 U(0) f(0) + S_2 (Uf)'(0) = 0,$$
$$T_1 U(1) f(1) + T_2 (Uf)'(1) = 0. \tag{2.2}$$

To prove the theorem we first need to show that the $2k$ new boundary functionals defined by 2.2 are linearly independent. Let $\mathfrak{L}_0$ (resp. $\mathfrak{L}_1$) be the minimal (resp. maximal) operator associated with $-D^2 + P$, and similarly let R_0 (resp. R_1) be the minimal (resp. maximal) operator associated with $(iD + Q)^2$. Recall that the graphs of these operators are Hilbert subspaces of $(\bigoplus^k L^2[0, 1]) \oplus (\bigoplus^k L^2[0, 1])$. For convenience we identify the operators with their graphs. The next lemma proves the desired linear independence.

(2.3) Lemma. *Let $U(X)$ be a twice continuously differentiable function from $[0, 1] \to \mathrm{Gl}[k, C]$. Let R be a subspace satisfying $R_0 \subset R \subset R_1$. If $\mathfrak{L} = U^{-1}RU$ is formally given by the same differential expression as $\mathfrak{L}_1$, then* $\dim(R_1 \ominus R) = \dim(\mathfrak{L}_1 \ominus \mathfrak{L})$.

Proof. The graph of $\mathfrak{L}$ is exactly $\{\{U^{-1}f, U^{-1}Rf\} | \{f, Rf\}$ is in the graph of $R\}$. Since U is smooth, U^{-1} maps $\mathfrak{D}(R_1)$ onto $\mathfrak{D}(\mathfrak{L}_1)$ and $\mathfrak{D}(R_0)$ into $\mathfrak{D}(\mathfrak{L}_0)$. Since $\dim(R_1 \ominus R) < \infty$ this forces $\dim(R_1 \ominus R) = \dim(\mathfrak{L} \ominus \mathfrak{L})$.

Proof of (2.1). To have $U^*[-D^2 + P]U = [iD + Q]^2 + C$, it suffices to have $-2U^* U' = 2iQ$. Since Q is self-adjoint we see that $(U^* U)' = 0$, so that any such $U(X)$ will be unitary if $U(0)$ is.

Recall that the determination of self-adjoint boundary conditions for $[iD + Q]^2$ requires knowledge of the matrix $B(x)$ associated with the semibilinear form of Green's formula (Coddington and Levinson [3] or Coddington and Dijksma [2]). In general if we consider $B(x)$ as an $n \times n$ matrix whose entries are $k \times k$ matrices, then

$$B_{lj} = \left\{ \begin{array}{ll} \displaystyle\sum_{m=l+j-1}^{n} (-1)^{m-j} \binom{m-j}{l-1} P_m^{(m-l-j-1)} & l + j \geqslant n + 1 \\ 0 & l + j < n + 1 \end{array} \right\},$$

where the differential operator has coefficients $P_m(x)$. In our case

$$B^{-1} = \begin{bmatrix} 0_k & -I_k \\ I_k & 2iQ \end{bmatrix}.$$

Denote by M and N the matrices

$$M = \begin{bmatrix} S_1 U(0) + S_2 U'(0) & S_2 U(0) \\ 0_k & 0_k \end{bmatrix}$$
$$= \begin{bmatrix} S_1 U(0) - iS_2 U(0) Q(0) & S_1 U(0) \\ 0_k & 0_k \end{bmatrix},$$
$$N = \begin{bmatrix} 0_k & 0_k \\ T_1 U(1) - iT_2 U(1) Q(1) & T_2 U(1) \end{bmatrix}.$$

The condition that (2.2) defines a self-adjoint formally given by $[iD + Q]^2$ is that rank $(M : N) = 2k$, which is guaranteed by (2.3), and that $MB^{-1}(0)M^* = NB^{-1}(1)N^* = 0_k$. Now

$$MB^{-1}(0)M^* = \begin{vmatrix} S_2U(0)\{S_1U(0)Q(0)\}^* & 0_k \\ -\{S_1U(0) - 3iS_2U(0)Q(0)\}\{S_2U(0)\}^* & \\ 0_k & 0_k \end{vmatrix}.$$

The condition $(MB^{-1}(0)M^*)_{11} = 0$ is exactly

$$-\tfrac{1}{2}\operatorname{Im}\{S_2S_1^*\} = \{S_2U(0)Q(0)U^*(0)S_2^*\}$$

if $U(0)$ is unitary. The analogous condition is equivalent to $NB^{-1}(1)N^* = 0$.

Thus the proof is done if we can find a twice continuously differentiable mapping U of $[0, 1]$ into the unitary $k \times k$ matrices, and a self-adjoint matrix-valued function Q such that $Q(0) = A_1$, $Q(1) = A_2$, $U' = -iUQ$, and $U(0)$ commutes with $Q(0)$ while $U(1)$ commutes with $Q(1)$. This is the content of the next lemma. □

(2.4) Lemma. *Given self-adjoint matrices A_1, A_2 there is a twice continuously differentiable self-adjoint matrix valued function $Q(x)$ on $[0, 1]$ such that in a neighborhood of zero $Q(x) = A_1$ and in a neighborhood of one $Q(x) = A_2$. Moreover the unitary matrix $U(x)$ defined by $U' = -iUQ$, $U(0) = I$ also satisfies the condition that $U(1)$ commutes with A_2.*

Proof. Let $0 < t_1 < t_2 < 1$. Define $Q(x)$ to be A_1 on $[0, t_1]$ and A_2 on $[t_2, 1]$. It suffices to find a twice continuously differentiable extension of Q to $[0, 1]$ so that $U(t_2) = I_k$.

To accomplish this we let $t_1 < t_3 < t_4 < t_2$ and find a real polynomial $p(x)$ with at least a triple zero at t_1, and such that $(p(x) - 1)$ has at least a triple zero at t_3. Then we can smoothly extend Q by defining $Q(x) = A_1(-p(x))$ for $t_1 \leqslant x \leqslant t_3$. We also have

$$U(t_3) = e^{-iA_1t_1}\exp\left(-iA_1\int_{t_1}^{t_3} p(s)\,ds\right),$$

so we impose the additional restriction that $\int_{t_1}^{t_3} p(s)\,ds = -1$. Now $U(t_3) = I$. If we make the analogous construction on $[t_4, t_2]$ and define $Q(x) = 0$ for $t_3 \leqslant x \leqslant t_4$, the problem is solved. □

(2.5) Corollary. *The hypotheses of (2.1) are satisfied if S_2 and T_2 are invertible, or if $k = 1$.*

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