EXPANSIONS ASSOCIATED WITH NON-SELF-ADJOINT BOUNDARY-VALUE PROBLEMS

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ABSTRACT. Eigenfunction expansions are given for a class of non-self-adjoint boundary-value problems. The results apply to systems of ordinary differential equations, some of whose coefficients may be arbitrary bounded Hilbert space operators.

Introduction. Given a differential operator \( L \) on \( \bigoplus^k L^2[0, 1] \) with compact resolvent, there are associated sequences of projections \( \{P_m\} \) obtained by integrating the resolvent around circles whose radii tend to infinity. The problem of obtaining an eigenfunction expansion associated with \( L \) is just the problem of showing that for each \( f \in \bigoplus^k L^2[0, 1] \), \( \lim_{m \to \infty} \|f - P_m f\| = 0 \). The usual technique for establishing such results involves an analysis of the kernel of the resolvent. If the operator is of high order, or one is dealing with a system of equations, this technique becomes unwieldy. We present an abstract approach to these problems which avoids the usual analysis of the resolvent kernel.

Denote the orthogonal sum \( L^2[0, 1] \oplus \cdots \oplus L^2[0, 1] \) (k summands) by \( \bigoplus^k L^2[0, 1] \). Let \( D = \frac{d}{dx} \). A trivial but useful fact is that the operator \( L = iD \) acting componentwise on \( \bigoplus^k L^2[0, 1] \) is self-adjoint if equipped with the domain of \( n \)-tuples \( (f_1, \ldots, f_n) \) of absolutely continuous functions whose derivatives are in \( L^2[0, 1] \), and such that \( (f_1, \ldots, f_n)(0) = (f_1, \ldots, f_n)(1) \). Moreover the spectrum of \( L \) is \( \{2\pi n | n \in \mathbb{Z}\} \), and each eigenvalue has multiplicity \( k \). We begin by showing that the distribution of eigenvalues for this example is typical for a large class of unbounded operators. Information about the eigenvalue distribution for higher order differential operators is easily obtained by use of the spectral mapping theorem.

Fixing notation, we let \( D(\mathcal{L}) \) be the domain of the operator \( \mathcal{L} \), and \( \sigma(\mathcal{L}) \) its spectrum. Denote by \( C \) the complex numbers and by \( \mathcal{L}^* \) the Hilbert space adjoint of the operator \( \mathcal{L} \). Finally we remark that many of the basic facts about ordinary differential operators can be found in Goldberg [5].

I. Eigenvalue comparisons and the expansion result. Let \( H \) be a Hilbert space with two Hilbert space norms, \( \| \cdot \|_1 \) and \( \| \cdot \|_2 \) which induce the same topology on \( H \). Assume that for all \( f \in H \), \( \|f\|_1 \leq \|f\|_2 \). Let \( \mathcal{L} \) and \( M \) be

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self-adjoint on $H$ with norms $\| \cdot \|_1$ and $\| \cdot \|_2$ respectively. We suppose that $\mathcal{L}$ and $M$ have the same domain and that their spectra are discrete and do not include zero. A well-known application of the closed graph theorem shows that there is a positive constant $\beta$ such that

$$\frac{1}{\beta} \| Mf \|_2 < \| \mathcal{L}f \|_1, \quad f \in \mathcal{D}(\mathcal{L}).$$

(†)

Define $\Pi_M((a, b))$ to be the number of eigenvalues of $M$ in the interval $(a, b)$, counted with multiplicity.

(1.1) LEMMA. $\Pi_M((a, b)) < \Pi_M((\beta a, \beta b))$.

PROOF. If not we can find $\psi$ such that $\| \psi \|_1 = 1$, and $\psi$ is in the span of the eigenfunctions of $\mathcal{L}$ with eigenvalues in $(-b, b)$, while being orthogonal to the span of the eigenfunctions of $M$ with eigenvalues in $(-\beta b, \beta b)$ with respect to the second inner product. Since $\| \psi \|_2 > 1$, we have $\| \mathcal{L}\psi \|_1 < b$ while $\| M\psi \|_2 > \beta b$. This contradicts (†), so establishes the result. □

Suppose that $\mathcal{L}$ and $\mathcal{E}$ are self-adjoint $n$th order ordinary differential operators on $\oplus^k L^2[0, 1]$, and that if $f$ is infinitely differentiable and compactly supported in $(0, 1)$, then $f \in \mathcal{D}(\mathcal{L}) \cap \mathcal{D}(\mathcal{E})$ and $\mathcal{L}f = \mathcal{E}f$. Moreover we suppose that $\mathcal{L}f = \sum_{j=0}^\infty A_j(x)D^j f$ for all $f$ as above, that $A_j$ is $j$-times continuously differentiable on $[0, 1]$, and that $\det(A_n(x)) \neq 0$ for $x \in [0, 1]$.

(1.2) LEMMA. For any open interval $\mathcal{I}$, $|\Pi_M(\mathcal{I}) - \Pi_M(\hat{\mathcal{I}})| < kn$.

PROOF. By adding a suitable real constant to $\mathcal{L}$ and $\mathcal{E}$ we can assume $\mathcal{I} = (-b, b)$, $b > 0$. Assume to the contrary that $\Pi_M((a, b)) - \Pi_M((a, b)) > kn + 1$. Since our differential operators fall into the regular case, there will be a function $\psi$ of unit norm which is a linear combination of eigenfunctions of $\mathcal{E}$ with eigenvalues in $(-b, b)$, $\psi \in \mathcal{D}(\mathcal{E}) \cap \mathcal{D}(\mathcal{E})$, and $\psi$ is orthogonal to the eigenfunctions of $\mathcal{L}$ with eigenvalues in $(-b, b)$. Thus we have $(\mathcal{L}\psi, \mathcal{E}\psi) = (\mathcal{E}\psi, \mathcal{E}\psi)$, but $\| \mathcal{L}\psi, \mathcal{E}\psi \| < b^2$ while $\| \mathcal{E}\psi, \mathcal{E}\psi \| > b^2$. This contradiction gives the result. □

Observe that if $\mathcal{L}$ is formally self-adjoint, regular, and of order $2n$, then the boundary conditions

$$f(0) = \cdots = f^{(n-1)}(0) = f(1) = \cdots = f^{(n-1)}(1) = 0$$

will make $\mathcal{L}$ a self-adjoint operator. Thus by using (1.1) and (1.2) and the remarks made earlier about $iD$ on $\oplus^k L^2[0, 1]$, we see that for any regular self-adjoint ordinary differential operator $\mathcal{L}$ of order $n$ there is a positive constant $c_1$ such that $\Pi_M((-b, b)) < c_1 b^{1/n}$ if $b > 1$.

Consider the circles centered at the origin with radii $m^n$, $m = 1, 2, \ldots$. To preserve the bound of $c_1 b^{1/n}$ eigenvalues in $(-b, b)$, no more than half of the annuli bounded by the circles of radius $(m + 1)^n$ and $m^n$, for $m = 1, 2, \ldots, r$ can contain more than $2c_1$ eigenvalues of $\mathcal{L}$. Consequently we have the following:
(1.3) Lemma. Let $\mathcal{L}$ be a regular nth order self-adjoint ordinary differential operator on $\oplus^k L^2[0, 1]$. Then there is an infinite collection of circles $S_m$ centered at zero with radii $R_m$ such that $m^n < R_m < (m + 1)^n$, and there is a positive constant $c_2$ such that the distance from $S_m$ to $\sigma(\mathcal{L})$ is at least $c_2m^{n-1}$.

Now let $\mathcal{L}$ be as above and define a formal operator $\Phi = \sum_{i=1}^{n-2} A_i D^i$, where $A_i$ is a bounded operator on $\oplus^k L^2[0, 1]$. Let $T = \mathcal{L} - \Phi$, and $R(z)$, $\tilde{R}(z)$ denote $(\mathcal{L} - zI)^{-1}$, $(T - zI)^{-1}$ respectively. The circles $S_m$ are as in (1.3).

(1.4) Theorem. Except for a discrete set of $z \in \mathbb{C}$, $\tilde{R}(z)$ exists. If $f \in \oplus^k L^2[0, 1]$ we have

$$f = \lim_{m \to \infty} \frac{1}{2\pi i} \int_{S_m} \tilde{R}(z)f\,dz.$$ 

Proof. Since $T - zI = [I - \Phi R(z)](\mathcal{L} - zI)$ whenever $R(z)$ exists, we have

$$\tilde{R}(z) = R(z)[I - \Phi R(z)]^{-1} \quad \text{if} \quad \|\Phi R(z)\| < 1.$$ 

Now

$$\|\Phi R(z)\| \leq (n-1) \sup_i \|A_i\| \sup_i \|D^i R(z)\|.$$ 

To estimate $\|D^i R(z)\|$ we use a lemma from Friedman, *Partial differential equations*, p. 19. There it is shown that

$$\|D^i u\|^2 < \varepsilon \|D^i u\|^2 + \frac{c}{\varepsilon^{i/(n-i)}} \|u\|^2 \quad \text{if} \quad i < n, \ 0 < \varepsilon < \varepsilon_0,$$

(††) where $\varepsilon_0$ and $c$ depend only on $n$.

Since $\|R(z)\| = 1/d(z, \sigma(\mathcal{L}))$, we have $\|R(z)\| \leq 1/c_2m^{n-1}$ if $z \in S_m$. By (††), if we let $|z| = R_m$ and $\varepsilon = (1/|z|)^{4/n}$ we have $\|D^i R(z)f\| \leq (c_3/R_m)|f|$, $i < n - 2$, where $c_3$ is a constant. Here we have used $\|D^i u\| \leq K(\|\mathcal{L}u\| + \|u\|)$, $K$ a constant. Thus for $z \in S_m$,

$$\|\Phi R(z)\| \leq (n-1) \sup_j \|A_j\| \frac{c_3}{m}.$$ 

(‡)

This establishes the existence of $\tilde{R}(z)$ for $m$ large. The discreteness of $\sigma(T)$ follows immediately.

Now for $z \in S_m$ and $f \in \oplus^k L^2[0, 1]$,

$$\frac{1}{2\pi i} \int_{S_m} \tilde{R}(z) f\,dz = \frac{1}{2\pi i} \int_{S_m} R(z)[I - \Phi R(z)]^{-1} f\,dz$$

$$= \frac{1}{2\pi i} \int_{S_m} R(z)f\,dz + \frac{1}{2\pi i} \int_{S_m} R(z)[I - \Phi R(z)]^{-1} \Phi R(z)f\,dz.$$ 

Since $\mathcal{L}$ is self-adjoint we know that $\lim_{m \to \infty} (1/2\pi i) \int_{S_m} R(z)f\,dz = f$. Thus we only need show that

$$\lim_{m \to \infty} \int_{S_m} R(z)[I - \Phi R(z)]^{-1} \Phi R(z)f\,dz = 0.$$
Let $\epsilon > 0$ and $f = \sum_{i=1}^{\infty} b_i x_i$, where $\hat{L} x_i = \lambda_i x_i$. Then we can find $j$ such that $\|f - \sum_{i=1}^{\infty} b_i x_i\| < \epsilon$. We have

$$
\int_{S_m} R(z) \left[ I - \Phi R(z) \right]^{-1} \Phi R(z) f \, dz
$$

$$
= \int_{S_m} R(z) \left[ I - \Phi R(z) \right]^{-1} \Phi R(z) \left\{ \sum_{i=1}^{j} b_i x_i \right\} \, dz
$$

$$
+ \int_{S_m} R(z) \left[ I - \Phi R(z) \right]^{-1} \Phi R(z) \left\{ f - \sum_{i=1}^{j} b_i x_i \right\} \, dz.
$$

This last integral is bounded in norm by $R_m(l/c^2 m^{-\alpha})(c^4/m)\epsilon$ by (\ref{eq:2}), where $c_4$ is a constant. To estimate the preceding integral we note that for $z \in S_m$ we have $\lim_{m \to \infty} \|R(z)\| = 0$, $\|R(z)\|^2$ is bounded, and

$$
\left\| \sum_{i=1}^{j} b_i \frac{\Phi x_i}{\lambda_i - z} \right\| = O \left( \frac{1}{|z|} \right).
$$

Thus the preceding integral goes to zero in norm as $m \to \infty$. This completes the proof.

The method of proof for this theorem is similar to one found in Clark [1]. He describes more detailed results when $\hat{L} = (iD)^n$ on $L^2[0, 1]$ with certain separated boundary conditions.

II. An application. Once we have an eigenfunction expansion for a differential operator, we also have an expansion for every operator similar to the one we started with. In some cases it is not difficult to identify a large collection of operators similar to those whose expansions we described in (1.4). That is the aim of this section.

Consider an operator $\hat{L}$ on $\Theta^k L^2[0, 1]$ formally given by $-D^2 + P$, where $P$ is a bounded operator on $\Theta^k L^2[0, 1]$. To define the domain of $\hat{L}$ we impose $2k$ separated boundary conditions on $f \in \Theta^k L^2[0, 1]$:

$$
S_1 f(0) + S_2 f'(0) = 0, \quad T_1 f(1) + T_2 f'(1) = 0.
$$

Here $S_1$, $S_2$, $T_1$, $T_2$ are $k \times k$ matrices such that the boundary functionals they define are linearly independent. Recall that for a $k \times k$ matrix $B$ we define $\text{Im} B = (B - B^*)/2i$.

\begin{equation}
(2.1) \text{Theorem. If there are self-adjoint $k \times k$ matrices } A_1, A_2 \text{ such that } -\frac{1}{2} \text{ Im}[S_2 S_2^*] = S_2 A_1 S_2^*, \text{ and } -\frac{1}{2} \text{ Im}[T_2 T_2^*] = T_2 A_2 T_2^*, \text{ then } \hat{L} \text{ is unitarily equivalent to a bounded perturbation of a second order self-adjoint regular differential operator on } \Theta^k L^2[0, 1].
\end{equation}

The plan is to show the existence of a smooth $k \times k$ unitary matrix valued function $U(x)$ such that $U^*[\gamma^2 - D^2 + P]U = [iD + Q]^2 + C$, where $Q$ is a bounded self-adjoint $k \times k$ matrix-valued function and $C$ is a bounded operator. The differential expression $[iD + Q]^2$ is formally self-adjoint, and
this new operator has defining boundary conditions
\[ S_1 U(0) f(0) + S_2 (Uf)'(0) = 0, \]
\[ T_1 U(1) f(1) + T_2 (Uf)'(1) = 0. \]  

(2.2)

To prove the theorem we first need to show that the 2k new boundary functionals defined by 2.2 are linearly independent. Let \( \mathcal{E}_0 \) (resp. \( \mathcal{E}_1 \)) be the minimal (resp. maximal) operator associated with \(-D^2 + P\), and similarly let \( R_0 \) (resp. \( R_1 \)) be the minimal (resp. maximal) operator associated with \((iD + Q)^2\). Recall that the graphs of these operators are Hilbert subspaces of \((\oplus^k L^2(0, 1)) \oplus (\oplus^k L^2(0, 1))\). For convenience we identify the operators with their graphs. The next lemma proves the desired linear independence.

(2.3) LEMMA. Let \( U(X) \) be a twice continuously differentiable function from \([0, 1] \rightarrow GL[k, C]\). Let \( R \) be a subspace satisfying \( R_0 \subset R \subset R_1 \). If \( \mathcal{E} = U^{-1}RU \) is formally given by the same differential expression as \( \mathcal{E}_1 \), then
\[ \dim(R_1 \ominus R) = \dim(\mathcal{E}_1 \ominus \mathcal{E}). \]

PROOF. The graph of \( \mathcal{E} \) is exactly \( \{(U^{-1}f, U^{-1}Rf) | (f, Rf) \text{ is in the graph of } R\} \). Since \( U \) is smooth, \( U^{-1} \) maps \( \oplus(R_1) \) onto \( \oplus(\mathcal{E}_1) \) and \( \oplus(R_0) \) into \( \oplus(\mathcal{E}_0) \). Since \( \dim(R_1 \ominus R) < \infty \) this forces
\[ \dim(R_1 \ominus R) = \dim(\mathcal{E}_1 \ominus \mathcal{E}). \]

PROOF OF (2.1). To have \( U^*[−D^2 + P]U = [iD + Q]^2 + C \), it suffices to have \(-2U^*U' = 2iQ\). Since \( Q \) is self-adjoint we see that \((U^*U)' = 0\), so that any such \( U(X) \) will be unitary if \( U(0) \) is.

Recall that the determination of self-adjoint boundary conditions for \([iD + Q]^2\) requires knowledge of the matrix \( B(x) \) associated with the semi-linear form of Green's formula (Coddington and Levinson [3] or Coddington and Dijksma [2]). In general if we consider \( B(x) \) as an \( n \times n \) matrix whose entries are \( k \times k \) matrices, then
\[ B_y = \begin{cases} \sum_{m=l+j-1}^{n} (-1)^{m-j} \binom{m-j}{l-1} p_m^{(m-l-j-1)} & l+j > n+1 \\ 0 & l+j < n+1 \end{cases}, \]
where the differential operator has coefficients \( p_m(x) \). In our case
\[ B^{-1} = \begin{bmatrix} 0_k & -I_k \\ I_k & 2iQ \end{bmatrix}. \]

Denote by \( M \) and \( N \) the matrices
\[ M = \begin{bmatrix} S_1 U(0) + S_2 U'(0) & S_2 U(0) \\ 0_k & 0_k \end{bmatrix} = \begin{bmatrix} S_1 U(0) - iS_2 U(0) Q(0) & S_1 U(0) \\ 0_k & 0_k \end{bmatrix}, \]
\[ N = \begin{bmatrix} 0_k & 0_k \\ T_1 U(1) - iT_2 U(1) Q(1) & T_2 U(1) \end{bmatrix}. \]
The condition that \((2.2)\) defines a self-adjoint formally given by \([iD + Q]^2\) is that \(\text{rank } (M : N) = 2k\), which is guaranteed by \((2.3)\), and that \(MB^{-1}(0)M^* = NB^{-1}(1)N^* = 0_k\). Now

\[
MB^{-1}(0)M^* = \begin{bmatrix}
S_2U(0)\{S_1U(0)Q(0)\}^* & 0_k \\
-S_1U(0) - 3iS_2U(0)Q(0)\{S_2U(0)\}^* & 0_k
\end{bmatrix}.
\]

The condition \((MB^{-1}(0)M^*)_{11} = 0\) is exactly

\[-\frac{1}{2}\text{Im}\{S_2S_1^*\} = \{S_2U(0)Q(0)U^*(0)S_1^*\}\]

if \(U(0)\) is unitary. The analogous condition is equivalent to \(NB^{-1}(1)N^* = 0\).

Thus the proof is done if we can find a twice continuously differentiable mapping \(U\) of \([0, 1]\) into the unitary \(k \times k\) matrices, and a self-adjoint matrix-valued function \(Q\) such that \(Q(0) = A_1\), \(Q(1) = A_2\), \(U' = -iUQ\), and \(U(0)\) commutes with \(Q(0)\) while \(U(1)\) commutes with \(Q(1)\). This is the content of the next lemma.

\((2.4)\) Lemma. Given self-adjoint matrices \(A_1, A_2\) there is a twice continuously differentiable self-adjoint matrix valued function \(Q(x)\) on \([0, 1]\) such that in a neighborhood of zero \(Q(x) = A_1\) and in a neighborhood of one \(Q(x) = A_2\). Moreover the unitary matrix \(U(x)\) defined by \(U' = -iUQ\), \(U(0) = I\) also satisfies the condition that \(U(1)\) commutes with \(A_2\).

Proof. Let \(0 < t_1 < t_2 < 1\). Define \(Q(x)\) to be \(A_1\) on \([0, t_1]\) and \(A_2\) on \([t_2, 1]\). It suffices to find a twice continuously differentiable extension of \(Q\) to \([0, 1]\) so that \(U(t_2) = I_k\).

To accomplish this we let \(t_1 < t_3 < t_4 < t_2\) and find a real polynomial \(p(x)\) with at least a triple zero at \(t_1\), and such that \((p(x) - 1)\) has at least a triple zero at \(t_3\). Then we can smoothly extend \(Q\) by defining \(Q(x) = A_1(-p(x))\) for \(t_1 < x < t_3\). We also have

\[
U(t_3) = e^{-iA_1t_1}\exp\left(-iA_1\int_{t_1}^{t_3} p(s) \, ds\right),
\]

so we impose the additional restriction that \(\int_{t_1}^{t_3} p(s) \, ds = -1\). Now \(U(t_3) = I\). If we make the analogous construction on \([t_4, t_2]\) and define \(Q(x) = 0\) for \(t_4 < x < t_2\), the problem is solved.

\((2.5)\) Corollary. The hypotheses of \((2.1)\) are satisfied if \(S_2\) and \(T_2\) are invertible, or if \(k = 1\).

Bibliography


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