ON WEYL FRACTIONAL CALCULUS

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ABSTRACT. The Weyl fractional calculus is applied in developing the Laplace transform of \( r^q f(t) \), for all values of \( q \). Also, a generalized Taylor's formula in Weyl fractional calculus is established. The results are then used in deriving a certain generating function for the \( H \)-function of Fox.

1. Introduction. Let \( A \) denote a class of good functions. By a good function \( f \), we mean (Miller [6, p. 82]) a function which is everywhere differentiable any number of times and if it and all of its derivatives are \( O(x^{-\nu}) \), for all \( \nu \) as \( x \) increases without limit.

We define the Weyl fractional derivatives of a function \( g(z) \) as follows: Let \( g \in A \), then

\[
 z^{D_q g(z)} = \frac{(-1)^q}{\Gamma(-q)} \int_0^\infty (u-z)^{-q-1} g(u) du, \quad \text{for } q < 0.
\]

For \( q > 0 \),

\[
 z^{D_q g(z)} = \frac{d^r}{dz^r} (z^{D_q^{-r} g(z)}),
\]

\( r \) being a positive integer such that \( r > q \).

We recall the definition of \( H \)-function of Fox [3, p. 408] in the form:

\[
 H(z) = H_{M,N}^{P,Q} \left[ \begin{array}{c}
 (a_1, \ldots, a_P), (b_1, \ldots, b_Q)
 \\
 (1, \ldots, 1, 1)
 \end{array} \right]
 = \frac{1}{2\pi i} \int_L \theta(s)z^s ds, \quad \omega = \sqrt{-1},
\]

where, for convenience,

\[
 \theta(s) = \frac{\prod_{i=1}^M \Gamma(b_i - \beta_i s) \prod_{i=1}^N \Gamma(1 - a_i + \alpha_i s)}{\prod_{i=M+1}^\infty \Gamma(1 - b_i + \beta_i s) \prod_{i=N+1}^\infty \Gamma(1 - a_i - \alpha_i s)},
\]

\( z \neq 0 \), and an empty product is interpreted as unity. The integers \( M, N, P, Q \), are such that \( 0 < M < Q, 0 < N < P \); the coefficients \( \alpha_i (i = 1, \ldots, P), \beta_i \)

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(i = 1, . . ., Q), are all assumed to be positive. L is a suitably chosen contour such that all the poles of \( \theta(s) \) are simple.

The \( H \)-function \( H(z) \) is a very general function and has for its particular cases a number of important special functions. For the conditions of existence of the function (3) and its various special cases, the paper of Gupta and Jain [4, pp. 596-600] may be referred to.

2. The Laplace transform of \( t^q f(t) \), for arbitrary \( q \).

**Theorem 1.** Let \( f(t) \) be such that

\[
g(p) = L[f(t); p] \quad \text{exists and belongs to } A,
\]

where \( L[f(t); p] \) denotes the Laplace transform of \( f(t) \). Then, for all \( q \),

\[
(-1)^q p^{D^q_s g(p)} = L[t^q f(t); p].
\]

**Proof.** For \( q < 0 \), we have in view of (1)

\[
(-1)^q p^{D^q_s g(p)} = \frac{1}{\Gamma(-q)} \int_{\rho}^{\infty} (u - p)^{-q-1} g(u) \, du
\]

Using Erdélyi [2, p. 202, (11)] to evaluate the inner integral on the R.H.S, we get

\[
(-1)^q p^{D^q_s g(p)} = L[t^q f(t); p], \quad \text{for } q < 0.
\]

For \( q > 0 \), invoking the definition (2), we can write:

\[
(-1)^q p^{D^q_s g(p)} = \frac{d^q}{dp^q} \left( \int_0^{\infty} e^{-p t} t^{-q-1} f(t) \, dt \right).
\]

Differentiating under the sign of integral, we again find that

\[
(-1)^q p^{D^q_s g(p)} = L[t^q f(t); p], \quad \text{for } q > 0.
\]

This completes the proof of the theorem.

3. The generalized Taylor’s formula. We prove the following theorem which may be regarded as a generalization of the familiar Taylor’s formula.

**Theorem 2.** Let

(i) \( a \) be a real number such that \( 0 < a < 1 \),

(ii) \( \eta \) be an arbitrary complex number and

(iii) \( g(p) \in A \).

Then

\[
g(p + t) = \sum_{n=-\infty}^{\infty} \frac{a t^{an+\eta}}{\Gamma(an + \eta + 1)} p^{D^{\eta+n s} g(p)}
\]

is valid for all \( t \) on the circle \( |t/p| = 1 \).
Proof. Since \( g \in A \), we may suppose \( g(z) = z^{-\lambda}f(z) \). Assume \( f(z) = \sum_{r=0}^{\infty} C_r z^r \).

The L.H.S. of (10) then gives \((p + t)^{-\lambda} \sum_{r=0}^{\infty} C_r (p + t)^r \). Now the R.H.S. of (10) is

\[
\sum_{n=-\infty}^{\infty} \frac{a t^{an+\eta}}{\Gamma(an + \eta + 1)} \sum_{r=0}^{\infty} C_r p^r \left( p^{-\lambda} \sum_{r=0}^{\infty} C_r p^r \right) = \sum_{n=-\infty}^{\infty} \frac{a t^{an+\eta}}{\Gamma(an + \eta + 1)} \sum_{r=0}^{\infty} C_r \cdot p^{a \eta + \eta} p^r \cdot p^{-\lambda r}.
\]

Since

\[ z^{D_{\beta}\xi} = \frac{(-1)^q \Gamma(\lambda + q)}{\Gamma(\lambda)} z^{-\lambda-q}, \]

valid for all \( q \), the R.H.S. of (10) after a little simplification reduces to

\[
p^{-\lambda} \sum_{r=0}^{\infty} C_r \cdot p^r \sum_{n=-\infty}^{\infty} a \left( \frac{\lambda - r + an + \eta \cdot 1}{an + \eta} \right) \left( \frac{-t}{p} \right)^{an+\eta} = p^{-\lambda} \sum_{r=0}^{\infty} C_r \cdot p^r \sum_{n=-\infty}^{\infty} a \left( \frac{-\lambda + r}{an + \eta} \right) \left( \frac{t}{p} \right)^{an+\eta},
\]

since \((-1)^q \Gamma(q) = \frac{(-n)^r}{r!} \).

The inner sum can be simplified by making use of the known formula of Osler [7, p. 46, (5.1)], viz.

\[ \sum_{n=-\infty}^{\infty} a \left( \frac{p}{an + \eta} \right)^{f^{an+\eta}} = (1 + t)^{p}, \quad \text{for } |t| = 1, \]

(12)

to get \( \sum_{r=0}^{\infty} C_r (p + t)^{-\lambda + r} \). This completes the proof of Theorem 2.

4. Generating functions. Suppose

\[ f(t) = t^{s-1} H_{M,Q}^{M,N} \left[ a \left( \frac{\alpha_i, \alpha_j}{1, p} \right) \right], \quad h > 0. \]

(13)

Then from (5) and Gupta [5, p. 83],

\[ g(p) = p^{-\sigma} H_{M+1,Q}^{M,N+1} \left[ a \left( 1 - \sigma, h, \frac{\alpha_i, \alpha_j}{1, p} \right) \right], \]

(14)

provided that

\[ \text{Re} \left[ \sigma + h(b_i/b_j) \right] > 0 \quad (i = 1, \ldots, M), \quad |\text{arg } a| < \frac{1}{2} K \pi, \]

where

\[ K = \sum_{i=1}^{M} (\beta_i) - \sum_{M+1}^{Q} (\beta_i) + \sum_{1}^{N} (\alpha_i) - \sum_{N+1}^{P} (\alpha_i) > 0. \]
Also, we have for arbitrary \( q \),
\[
Z^{D_{z}^{\lambda}H_{P,Q}^{M,N}} \left[ a_{z}^{-h} \begin{pmatrix} (a_{i}, \alpha_{i})_{1,p} \\ (b_{i}, \beta_{i})_{1,Q} \end{pmatrix} \right] = (-1)^{q}Z^{\lambda-\eta H_{P+1,Q}^{M,N+1}} \left[ a_{z}^{-h} \begin{pmatrix} (1 + \lambda - q, h), (a_{i}, \alpha_{i})_{1,p} \\ (b_{i}, \beta_{i})_{1,Q}, (1 + \lambda, h) \end{pmatrix} \right],
\]
\[(14a)\]
provided that
\[
\text{Re}\left[ -\lambda + h \frac{\beta}{\eta} \right] > 0 \quad (i = 1, \ldots, M), \quad h > 0, \quad |\arg a| < \frac{1}{2} K \pi,
\]
where \( K = \sum_{i=1}^{M} (\beta_{i}) - \sum_{i=M+1}^{Q} (\beta_{i}) + \sum_{j=1}^{N} (\alpha_{j}) - \sum_{j=N+1}^{P} (\alpha_{j}) > 0. \)
\[(14a)\]
can be established easily by using the contour integral representation (3) and (11).

Substituting \( f(t) \) and \( g(p) \) from (13) and (14) respectively in the generalized Taylor’s formula (10) and using (14a) appropriately, we get
\[
(p + t)^{-\sigma}H_{P+1,Q}^{M,N+1} \left[ d'p + t \right]^{-h} \begin{pmatrix} (1 - \sigma, h), (a_{i}, \alpha_{i})_{1,p} \\ (b_{i}, \beta_{i})_{1,Q} \end{pmatrix} = p^{-\sigma} \sum_{n=0}^{\infty} \frac{a(-t/p)^{an+\eta}}{\Gamma(an + \eta + 1)}
\cdot H_{P+1,Q}^{M,N+1} \left[ d'p^{-h} \begin{pmatrix} (1 - \sigma - an - \eta, h), (a_{i}, \alpha_{i})_{1,p} \\ (b_{i}, \beta_{i})_{1,Q} \end{pmatrix} \right].
\]
\[(15)\]
Putting \( p = 1, \sigma = 1 - a_{1}, h = a_{1} \) and adjusting the other parameters, we get the generating function
\[
(1 + t)^{a_{1}-1}H_{P,Q}^{M,N} \left[ z(1 + t)^{-a_{1}} \begin{pmatrix} (a_{i}, \alpha_{i})_{1,p} \\ (b_{i}, \beta_{i})_{1,Q} \end{pmatrix} \right] = \sum_{n=-\infty}^{\infty} \frac{a(-t)^{an+\eta}}{\Gamma(an + \eta + 1)} H_{P,Q}^{M,N} \left[ z \begin{pmatrix} (a_{i} - an - \eta, \alpha_{i}), (a_{i}, \alpha_{i})_{2,p} \\ (b_{i}, \beta_{i})_{1,Q} \end{pmatrix} \right].
\]
\[(16)\]
Obviously for \( \eta = 0 \) and \( a = 1 \), (16) corresponds to a known result of Anandani [1, p. 6, (2.1)].

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