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ON THE EXISTENCE OF FUNDAMENTAL SOLUTIONS OF CERTAIN BOUNDARY VALUE PROBLEMS

R. S. PATHAK

ABSTRACT. An existence theorem for fundamental solutions of a large class of boundary value problems involving the Bessel type differential operator S_μ is proved. The problem includes the Cauchy problem of the differential operator S_μ as a special case.

1. Barros-Neto [1] has given fundamental solutions of the boundary value problem in a half space with constant coefficients. The problem can be described in brief as follows:

Let $P(D, D_t)$ be a hypoelliptic partial differential operator and let $Q_1(D, D_t), \ldots, Q_m(D, D_t)$ be given partial differential operators in R^{n+1} with constant coefficients, where

$$D = (D_1, \ldots, D_n), \quad D_j = (1/i)(\partial/\partial x_j), \quad D_t = (1/i)(\partial/\partial t),$$

and m is the number of roots with positive imaginary parts of the characteristic polynomial in τ, $P(\xi, \tau) = 0$. Then the solution of the boundary value problem defined by the operators $(P(D, D_t), Q_1(D, D_t), \ldots, Q_m(D, D_t))$ in the half space $R_+^{n+1} = \{(x, t): x \in R^n, t > 0\}$ are certain tempered distributions.

In this paper we are interested in the solutions of another class of boundary value problems. Let S_μ denote the differential operator $d^2/dx^2 + (1 - 4\mu^2)/4x^2$, $\mu \geqslant -\frac{1}{2}$. Then by using the theory of distributional Hankel transforms ([6] and [9]), we show that the solutions of the boundary value problem defined by $(P(S_\mu, D_t), Q_1(S_\mu, D_t), \ldots, Q_m(S_\mu, D_t))$ are certain generalized functions belonging to H'_μ [6].

2. The main results are contained in the following:

THEOREM. *There are distributions $K(x, t), K_1(x, t), \ldots, K_m(x, t)$ in $H'_\mu(R_+^{n+1})$ such that $K, K_1, \ldots, K_m$ are infinitely differentiable functions in R_+^{n+1} which can be extended to infinitely differentiable functions in $\overline{R_+^{n+1}} - \{0\}$. If we keep the same notation for the extended functions, then $K(x, t)$ satisfies, in the sense of distributions, the boundary problem*

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$$P(S_\mu, D_t)K = \delta(x-1) \otimes \delta(t) - \alpha(x) \otimes \delta(t) \quad in\ \overline{R_+^{n+1}},$$

$$Q_\nu(S_\mu, D_t)K\big|_{R_0^n} = 0, \qquad 1 \leqslant \nu \leqslant m, \tag{1}$$

while every $K_l(x, t)$ satisfies

$$P(S_\mu, D_t)K_l = 0 \quad in\ R_+^{n+1},$$

$$Q_\nu(S_\mu, D_t)K_l\big|_{R_0^n} = \delta_{\nu,l}(\delta(x-1) - \alpha(x)), \qquad 1 \leqslant \nu \leqslant m, \tag{2}$$

where $\delta_{\nu,l}$ is the Kronecker delta, δ is the Dirac measure, $\alpha(x) \in H_\mu(R^n)$ and

$$Q_\nu K\big|_{R_0^n} = \lim_{t \to 0^+} Q_\nu K(x, t)$$

the limit being taken in $H_\mu'(R^n)$.

Proof. For each $\xi \in R^n$, let $\tau_1(\xi), \ldots, \tau_m(\xi)$ be the roots of the characteristic polynomial $P(\xi, \tau) = 0$ with positive imaginary parts. Let

$$K_\xi(\tau) = \prod_{j=1}^{m} (\tau - \tau_j(\xi))$$

and define the characteristic function $C(\xi)$ by

$$C(\xi) = R(K_\xi; Q_1, \ldots, Q_m) = \frac{\det Q_\nu(\xi, \tau_j(\xi))}{\prod_{k<j}(\tau_j(\xi) - \tau_k(\xi))},$$

$$1 \leqslant \nu \leqslant m,\ 1 \leqslant j \leqslant m. \tag{3}$$

Then $C(\xi)$ is a polynomial in ξ and according to Hörmander [4] there exists a constant $M > 0$ such that for all $\xi \in R^n$ with

$$|\xi| > M, \quad C(\xi) \neq 0.$$

Next, for $\nu = 1, 2, \ldots, m$, $t \geqslant 0$ and for all ξ satisfying $|\xi| > M$ define the function $H_\nu(\xi, t)$ by

$$H_\nu(\xi, t) = \frac{R(K_\xi; Q_1(\xi, \tau(\xi)), \ldots, e^{it\tau(\xi)}, \ldots, Q_m(\xi, \tau(\xi))}{C(\xi)} \tag{4}$$

where

$$R\big(K_\xi; Q_1(\xi, \tau(\xi)), \ldots, e^{it\tau(\xi)}, \ldots, Q_m(\xi, \tau(\xi))\big)$$

indicates that in the determinant appearing in (3) the νth row has been replaced by $(e^{it\tau_1(\xi)}, \ldots, e^{it\tau_m(\xi)})$. It is easily seen that $H_\nu(\xi, t)$, $1 \leqslant \nu \leqslant m$, is a solution of the initial value problem

$$P(\xi, D_t)H_\nu(\xi, t) = 0, \tag{5}$$

$$Q_l(\xi, D_t)H_\nu(\xi, 0) = \delta_{l,\nu}, \qquad 1 \leqslant l \leqslant m.$$

Now, let us introduce a function $G_0(\xi, t)$ defined by

$$G_0(\xi, t) = \frac{1}{2\pi} \int_{-\infty}^{\infty} \frac{e^{it\tau}}{P(\xi, \tau)}\, d\tau \tag{6}$$

the integral being absolutely convergent if the degree of $P(\xi, \tau)$ in τ is $\geqslant 2$

and convergent if it is equal to 1. Obviously,

$$P(\xi, D_t)G_0(\xi, t) = \delta(t).$$

Next, for all $|\xi| > M$, set

$$G(\xi, t) = G_0(\xi, t) - \sum_{\nu=1}^{m} (Q_\nu(\xi, D_t)G_0)(\xi, 0)H_\nu(\xi, t). \tag{7}$$

Then $G(\xi, t)$ is a solution of the initial value problem

$$\begin{aligned} P(\xi, D_t)G(\xi, t) &= \delta(t), \\ Q_l(\xi, D_t)G(\xi, 0) &= 0, \qquad 1 < l < m, \end{aligned} \tag{8}$$

for all $|\xi| > M$. Assume that $M > 0$ is such that when $|\xi| > M$, G and H_ν, $1 < \nu < m$, are well defined. We define a new function $\chi(\xi) \in C_c^\infty(R^n)$ by

$$\begin{aligned} \chi(\xi) &= 1, \qquad |\xi| < M, \\ &= 0, \qquad |\xi| > M + 1. \end{aligned}$$

Then following the technique of Barros-Neto [1] and using the fact that $|y_i^{1/2}J_\mu(y_i)| < C$ for all $y_i > 0$ and $\mu > -\frac{1}{2}$, we can prove that

$$(1 - \chi(y))G(-y^2, t) \prod_{i=1}^{n} y_i^{1/2}J_\mu(y_i)$$

and

$$(1 - \chi(y))H_\nu(-y^2, t) \prod_{i=1}^{n} y_i^{1/2}J_\mu(y_i)$$

are tempered distributions in R^n which are concentrated on $\{y\colon |y| > M\}$ and therefore by the n-dimensional analogue of [9, Problem 5.2.3, p. 134] these are elements of H_μ'. Hence, we can define their generalized Hankel transforms h_μ'. Since for $\mu > -\frac{1}{2}$, h_μ' is an automorphism on H_μ' [6, p. 430] the inverse Hankel transforms can also be defined. Consequently, we are justified in defining $K(x, t)$ and $K_\nu(x, t)$ by

$$K(x, t) \triangleq h'_{\mu,x}\left((1 - \chi(y))G(-y^2, t) \prod_{i=1}^{n} y_i^{1/2}J_\mu(y_i)\right) \tag{9}$$

and

$$K_\nu(x, t) \triangleq h'_{\mu,x}\left((1 - \chi(y))H_\nu(-y^2, t) \prod_{i=1}^{n} y_i^{1/2}J_\mu(y_i)\right), \qquad 1 < \nu < m. \tag{10}$$

These are also elements of H_μ'. Following the technique of Barros-Neto [1] it can be proved that these kernels satisfy (1) and (2) and they are C^∞-functions in $R_+^{n+1} - \{0\}$.

Now, using the n-dimensional analogue of the results [9, pp. 143, 144]

$$h'_\mu(-y^2 f) = S_\mu h'_\mu f,$$

$$h'_\mu \delta(x-1) = \prod_{i=1}^{n} y_i^{1/2} J_\mu(y_i),$$

$$h'_\mu\left[\prod_{i=1}^{n} x_i^{1/2} J_\mu(x_i)\right] = \delta(y-1),$$

and operating formally we have

$$\begin{aligned} P(S_\mu, D_t)K &= h'_\mu\left((1-\chi(y))P(-y^2, D_t)G(-y^2, t)\prod_{i=1}^{n} y_i^{1/2}J_\mu(y_i)\right) \\ &= h'_\mu\left((1-\chi(y))\prod_{i=1}^{n} y_i^{1/2}J_\mu(y_i)\right) \otimes \delta(t) \\ &= \delta(x-1)\otimes\delta(t) - \alpha(x)\otimes\delta(t) \end{aligned}$$

and

$$\lim_{t\to 0^+} Q_\nu(S_\mu, D_t)K$$
$$= h'_\mu\big((1-\chi(y))P(-y^2, D_t)G(-y^2, 0)\big) = 0, \qquad 1 \leqslant \nu \leqslant m.$$

Similarly we can derive the results (2).

Our theorem will be proved by justifying the above formal manipulations. For this, we need merely to prove the above limiting operations in H'_μ, which is the crucial part of the work. Let $\phi(x) \in H_\mu$; then we have to prove that

$$\langle Q_\nu(S_\mu, D_t)K_l(x,t), \phi(x)\rangle \to \langle Q_\nu(S_\mu, D_t)K_l(x,0), \phi(x)\rangle$$
$$(1 < l < m) \quad (11)$$

and

$$\langle Q_\nu(S_\mu, D_t)K(x,t), \phi(x)\rangle \to \langle Q_\nu(S_\mu, D_t)K(x,0), \phi(x)\rangle \qquad (12)$$

as $t \to 0^+$.

Now, in view of the definition (10), we have

$$\begin{aligned} &\langle Q_\nu(S_\mu, D_t)K_l(x,t), \phi(x)\rangle \\ &\quad = \left\langle Q_\nu(S_\mu, D_t)h'_\mu\left[(1-\chi(y))H_l(-y^2, t)\prod_{i=1}^{n} y_i^{1/2}J_\mu(y_i)\right], \phi(x)\right\rangle \\ &\quad = \left\langle (1-\chi(y))Q(-y^2, \tau)H_l(-y^2, t)\prod_{i=1}^{n} y_i^{1/2}J_\mu(y_i), \Phi(y)\right\rangle \end{aligned} \qquad (13)$$

where $\Phi(y) = h_\mu[\phi(x)]$.

Expanding the determinant by means of which $H_l(\xi, t)$ has been defined, we have

$$H_l(\xi, t) = \sum_{r=1}^{m} e^{it\tau_r(\xi)} A_\tau^r(\xi)/C(\xi),$$

where $A_\tau^r(\xi)$ is a polynomial in ξ. Hence the expression (13) can be written as

$$\sum_{r=1}^{m} \left\langle (1 - \chi(y)) Q_\nu(-y^2, \tau) \prod_{i=1}^{n} y_i^{1/2} J_\mu(y_i) A_\tau^r(-y^2)/C(-y^2), e^{it\tau_r(-y^2)}\Phi(y) \right\rangle.$$

In the following we shall show that, as $t \to 0^+$, $e^{it\tau_r(-y^2)}\Phi(y)$ converges in H_μ to $\Phi(y)$. Then the limiting operation (11) will follow from this result and the fact that

$$(1 - \chi(y)) Q_\nu(-y^2, \tau) \prod_{i=1}^{n} y_i^{1/2} J_\mu(y_i) \frac{A_\tau^r(-y^2)}{C(-y^2)} \in H_\mu'.$$

According to an n-dimensional analogue of [9, see 5.3] our above assertion will be established as soon as we prove that as $t \to 0^+$,

$$\frac{e^{it\tau_r(-y^2)}}{1 + y^{n_\nu}} \to \frac{1}{1 + y^{n_\nu}} \tag{14}$$

for some positive integer n_ν and, for each positive integer ν,

$$\frac{\left(y^{-1}D_y\right)^\nu}{1 + y^{n_\nu}} e^{it\tau_r(-y^2)} \to 0 \quad \text{as } t \to 0^+ \tag{15}$$

uniformly for all $y \in R^n - \{0\}$.

Notice that

$$\left(y^{-1}D_y\right)^\nu e^{it\tau_r(-y^2)} = 2^\nu \left[D_u^\nu e^{it\tau_r(-u)}\right]_{u=y^2}.$$

The differentiation of $e^{it\tau_r(-u)}$ with respect to u yields a finite number of terms which are of the form

$$A_r t^p \tau_r^{(q)}(-u) e^{it\tau_r(-u)}.$$

Here $\tau_r^{(q)}$ denotes $(d/dy)^q \tau_r(-y)$. Since all the roots of $P(\xi, \tau) = 0$ satisfy the inequality [2, p. 518]

$$|\tau(\xi)| \leqslant A\left(|\xi|^B + 1\right), \tag{16}$$

and by Lemma 4 of [2],

$$|D_\xi^q \tau(\xi)| \leqslant A_q' |\xi|^{B+|q|} \left(\sum_{j=1}^{n} |\xi_j|^{1/d_j}\right)^{-|q|}$$

for all $\xi \in R^n$ such that $\sum_{j=1}^n |\xi_j|^{1/d_j} \geqslant M + 1$, we have

$$\left|t^p \tau_r^{(q)} e^{it\tau_r(-u)}\right| \leqslant C t^p |u|^{C'}$$

for certain finite constants C and C'. Therefore, choosing $n_\nu > 2C'$, we have

$$\frac{\left(y^{-1}D_y\right)^\nu}{1 + y^{n_\nu}} e^{it\tau_r(-y^2)} \to 0 \quad \text{as } t \to 0,$$

uniformly for all $y \in R^n - \{0\}$.

Further to prove (14), we first note that, given $\varepsilon > 0$, there exists an $R < \infty$, such that, for all $y > R$ and $t \geqslant 0$,

$$0 \leqslant \frac{|1 - e^{it\tau_r(-y^2)}|}{1 + y^{n_\nu}} \leqslant \frac{1 + e^{-t|\operatorname{Im}\tau_r(-y^2)|}}{1 + R^{n_\nu}} \leqslant \frac{2}{1 + R^{n_\nu}} < \varepsilon.$$

Having fixed R this way, we restrict y to $0 < y \leqslant R$ and then write

$$0 \leqslant \frac{|1 - e^{it\tau_r(-y^2)}|}{1 + y^{n_\nu}} \leqslant \frac{\int_0^t |\tau_r(-y^2)| e^{-s \operatorname{Im}\tau_r(-y^2)}\, ds}{1 + y^{n_\nu}}$$

$$\leqslant \frac{t|\tau_r(-y^2)|}{1 + y^{n_\nu}} \leqslant C\frac{t(|y|^{2d} + 1)}{1 + y^{n_\nu}}$$

by (16). Choosing $n_\nu > 2d$, we see that the last expression tends to zero as $t \to 0^+$ uniformly for all $y \in R^n$. Therefore, there exists a $T > 0$ such that, for all $0 < t < T$,

$$0 \leqslant \frac{|1 - e^{it\tau_r(-y^2)}|}{1 + y^{n_\nu}} \leqslant \varepsilon, \qquad y \in R^n.$$

Since ε is arbitrary our assertion is proved.

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Department of Mathematics, Banaras Hindu University, Varanasi 221005, India