A NOTE ON GENERATING RELATIONS FOR LAURICELLA’S FUNCTION OF SEVERAL VARIABLES

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Abstract. In this paper we have derived generating relations for Lauricella’s function by using the operator $T_k = x(k + xD)$ and the operational relations involving this operator. Some recent results of Srivastava [3], Srivastava and Carlitz [4], Sharma and Abiodun [1] have been conveniently obtained by this method as well as some hitherto unknown results established.

1. Introduction. Srivastava [2], [3] considered the function

$$f(z_1, \ldots, z_r) = \sum_{k_1, \ldots, k_r=0}^{\infty} C_{k_1, \ldots, k_r} \prod_{j=1}^r \frac{(z_j)^{k_j}}{k_j!}, \quad (1.1)$$

where $C_{k_1, \ldots, k_r}$ are arbitrary complex constants, and derived the generating relation (2.2). Considering the function in (1.1), Carlitz and Srivastava [4] obtained several generating functions for Lauricella’s function of several variables.

Agarwal [8] and Sharma and Abiodun [1] considered the general $G$-function (4.1) and Sharma and Abiodun [1] derived a generating relation for it.

In this paper, making use of operational methods, we propose to derive results of Srivastava [3], Carlitz and Srivastava [4], Sharma and Abiodun [1] and other results directly by making use of the operator formulas (2.1), (3.1), (3.4), (3.6) and (3.8).

In §2 we obtain a generating relation for $f(z_1, \ldots, z_r)$ defined in (1.1) and, in §3, we derive various generating functions for Lauricella’s function of several variables. In §4 we consider the general $G$-function and, using (2.1), we derive directly the result due to Sharma.

The series occurring in the paper may be thought of as formal power series.

2. Generating relations for functions of several variables. In what follows we shall make use of the following Mittal [7] operational generating formula

$$\sum_{n=0}^{\infty} \frac{z^n}{n!} T_{a+1+(m-1)v}^n(x) = \frac{(1+v)^{a+1}}{1-(m-1)v} f[x(1+v)] \quad (2.1)$$

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where \( v = xz(1 + v)^{m} \), \( m \) being constant and \( f(x) \) admits a formal power series in \( x \) and \( T_k \equiv x(k + xD), D = d/dx \).

Assuming that the operator \( T_k \) operates on \( t \) alone and putting \( a = \alpha, m = \beta + 1 \) and

\[
f(x) \equiv \int [x_1 (-y^{-\beta_1 + 1})^{m_1}, \ldots, x_r (y^{-\beta_1})^{m_r}] \text{ in (2.1)}
\]

we immediately get

\[
\sum_{n=0}^{\infty} \left( \frac{\alpha + (\beta + 1)n}{n} \right) t^n \sum_{k_1, \ldots, k_r = 0}^{n} \frac{(-n) \Sigma \sigma_{m_k}^2 - \Sigma \sigma_{m_k}}{(1 + \alpha + \beta n) \Sigma \sigma_{m_k}}
\]

\[
\cdot C(k_1, \ldots, k_r) t^{\beta \Sigma \sigma_{m_k} + \gamma - \beta v \sigma_{m_k}} \frac{x_1^{k_1}}{k_1!} \ldots \frac{x_r^{k_r}}{k_r!}
\]

\[
= (1 + v)^{a+1}(1 - \beta_0)^{-1} f \left\{ x_1 \left[ -y^{-m_1 \beta} \left\{ t^{\beta+1}(1 + v)^{\beta+1} \right\}^{m_1} \right], \ldots, x_r \left[ -y^{-m_r \beta} \left\{ t^{\beta+1}(1 + v)^{\beta+1} \right\}^{m_r} \right] \right\}.
\] (2.2)

where \( v = tz(1 + v)^{\beta+1} \). Now putting \( z = 1 \) and \( y = t \) in (2.2) we get

\[
\sum_{n=0}^{\infty} \left( \frac{\alpha + (\beta + 1)n}{n} \right) t^n \sum_{k_1, \ldots, k_r = 0}^{n} \frac{(-n) \Sigma \sigma_{m_k}^2 - \Sigma \sigma_{m_k}}{(1 + \alpha + \beta n) \Sigma \sigma_{m_k}}
\]

\[
\cdot C(k_1, \ldots, k_r) t^{\beta \Sigma \sigma_{m_k} + \gamma - \beta v \sigma_{m_k}} \frac{x_1^{k_1}}{k_1!} \ldots \frac{x_r^{k_r}}{k_r!}
\]

\[
= (1 + v)^{a+1}(1 - \beta_0)^{-1} f \left\{ x_1 \left[ -y^{-m_1 \beta} \left\{ t^{\beta+1}(1 + v)^{\beta+1} \right\}^{m_1} \right], \ldots, x_r \left[ -y^{-m_r \beta} \left\{ t^{\beta+1}(1 + v)^{\beta+1} \right\}^{m_r} \right] \right\}.
\] (2.3)

where \( v = t \lambda(1 + v)^{\beta+1} \), which is due to Srivastava [2].

3. Lauricella's function of several variables. In this section we shall use the Mittal [6] operational formulae

\[
\sum_{n=0}^{\infty} \frac{1}{n!} T_{a+1}^n \{ f(x) \} = (1 - x)^{-a-1} f \left[ \frac{x}{1-x} \right]
\] (3.1)

where \( f(x) \) admits a formal power series in \( x \). Putting \( a = \lambda - 1 \), and

\[
f(x) \equiv F_B[\lambda, m_i]: (\beta_i, \varphi_i); (\gamma, \psi_i); x_1(-t)^{m_1}, \ldots, x_r(-t)^{m_r}]
\]

in (3.1) and assuming that \( T_k \) operates on \( t \) alone, we get

\[
\sum_{n=0}^{\infty} \frac{(\lambda)^n}{n!} F_B[(-n, m_i): (\beta_i, \varphi_i); (\gamma, \psi_i); x_1, \ldots, x_r] t^n
\]

\[
= (1 - t)^{-\lambda} F_B[\lambda, m_i]: (\beta_i, \varphi_i); (\gamma, \psi_i); x_1 \left( \frac{t}{t - 1} \right)^{m_1}, \ldots, x_r \left( \frac{t}{t - 1} \right)^{m_r}]
\] (3.2)
which is due to Carlitz and Srivastava [4]. Carlitz and Srivastava [4] also proved that

\[ \sum_{n=0}^{\infty} \frac{(\lambda)_n(\gamma)_n t^n}{n! (\mu)_n} F_{\beta}^{(\gamma)} \left[ (-n, m_i); (\beta_i, \varphi_i); (\gamma, m_i); x_1, \ldots, x_r \right] \\
= (1 - t)^{-\lambda} F_{\beta}^{(\gamma + 1)} \left[ (\lambda, m_i, 1); (\beta, \varphi_i), (\mu - \gamma, 1); (\mu; m_i, 1) \right] \\
x_1 \left( \frac{t}{t - 1} \right)^{m_1}, \ldots, x_r \left( \frac{t}{t - 1} \right)^{m_r}, \frac{t}{t - 1} \right]. \quad (3.3) \]

Putting \( a = \lambda - 1 \) and

\[ f(x) = F_{\beta}^{(\gamma + 1)} \left[ (\lambda, m_i, 1); (\beta_i, \varphi_i), (\mu - \gamma, 1); (\mu; m_i, 1) \right] \\
x_1 (-t)^{m_1}, \ldots, x_r (-t)^{m_r}, (-t) \]

in (3.1) and assuming that \( T_k \)-operates on \( t \) alone, and using Euler's transformation, the result in (3.3) follows. Again by making use of the Mittal [6] operational formula

\[ \sum_{n=0}^{\infty} \frac{1}{n!} T_{a+n}^n \{ f(x) \} = (1 - 4x)^{-1/2} \left[ \frac{2}{1 + (1 - 4x)^{1/2}} \right]^{-a+1} \]

Putting \( a = \lambda \) and

\[ f(x) = F_{\beta}^{(\gamma)} \left[ (\lambda, m_i); (\beta_i, \varphi_i); (\gamma, \psi_i); x_1 t^{m_1}, \ldots, x_r t^{m_r} \right] \]

in (3.4) and assuming that \( T_k \)-operates on \( t \) alone, we get

\[ \sum_{n=0}^{\infty} \frac{(\lambda)_2 t^n}{n! (\lambda)_n} F_{\beta; 0, 1, \ldots, 1}^{(\gamma)} \left[ (\lambda, m_i), (-n, m_i); (\beta_i, \varphi_i); x_1, \ldots, x_r \right] \\
= (1 - 4t)^{-1/2} \left[ \frac{2}{1 + (1 - 4t)^{1/2}} \right]^{\lambda - 1} \]

\[ \cdot F_{\beta}^{(\gamma)} \left[ (\lambda, m_i); (\beta, \varphi_i); (\gamma, \psi_i); \\
x_1 \left( \frac{2t}{1 + (1 - 4t)^{1/2}} \right)^{m_1}, \ldots, x_r \left( \frac{2t}{1 + (1 - 4t)^{1/2}} \right)^{m_r} \right]. \quad (3.5) \]

Putting \( \lambda = \gamma \) and \( \psi_i = m_i \) in (3.5) we get the following generating relation for
the Lauricella function

\[
\sum_{n=0}^{\infty} \frac{(\lambda)_{2n}}{n! (\lambda)_n} F^\mathcal{G}_{\mathcal{Y}}\left[\begin{matrix}(\lambda, n): (\beta, \varphi); (1 - \lambda - 2n, m); x_1, \ldots, x_r
\end{matrix}\right] = (1 - 4t)^{1/2}\frac{2}{1 + (1 - 4t)^{1/2}}^{\lambda-1}
\]

Again by making use of the Mittal [6] operational formula

\[
\sum_{n=0}^{\infty} \frac{1}{n!} T_a^n \{f(x)\} = (1 + x)^{a-1} f[x(1 + x)].
\]

Proceeding as above we get the generating relation

\[
\sum_{n=0}^{\infty} (-1)^a \frac{(1 - \lambda)_n}{n!} \left(\frac{\lambda}{2}\right)^{\mathcal{Z}_{1:0, \ldots, 0}} \left[\begin{matrix}(\lambda, m), (\lambda, m)(-n, m); \\
(\beta, \varphi); x_1, \ldots, x_r \\
(\gamma, \psi); (\lambda - n, m)
\end{matrix}\right]
\]

\[
= (1 + t)^{\lambda-1} F^\mathcal{G}_{\mathcal{Y}}\left[\begin{matrix}(\lambda, m): (\beta, \varphi); (\gamma, \psi); \\
x_1 \{t(1 + t)^{m_1}, \ldots, x_r \{t(1 + t)^{m_r}\}
\end{matrix}\right].
\]

Next using the Mittal [6] operational formula,

\[
\sum_{n=0}^{\infty} \frac{1}{n!} T_{a-2n} \{f(x)\} = (1 + 4x)^{-1/2} \left[\begin{matrix}2 \\
1 + (1 + 4x)^{1/2}
\end{matrix}\right]^{-a} f\left[\frac{x(1 + (1 + 4x)^{1/2})}{2}\right].
\]

Putting \(a = \lambda\), and

\[
f(x) = F^\mathcal{G}_{\mathcal{Y}}\left[\begin{matrix}(\lambda, m): (\beta, \varphi); (\gamma, \psi); x_1 \left(\frac{2}{3} t\right)^{m_1}, \ldots, x_r \left(\frac{2}{3} t\right)^{m_r}
\end{matrix}\right]
\]
in (3.9) and assuming that $T_k$-operates on $t$ alone we get
\[
\sum_{n=0}^{\infty} (-1)^n \frac{(1-\lambda)_{2n}}{(1-\lambda)_n} \frac{t^n}{n!}
\]
\[
\cdot F_{4;0,\ldots,0}^{4;1,\ldots,1,0} \left\{ (-n, m_i), (\lambda, m_i) \left( \frac{\lambda - n + 1}{2}, m_i \right), \left( \beta_i, \varphi_i \right); (\gamma, \psi_i) \right\}
\]
\[
\cdot \cdot\cdot F_B^{\vartheta} \left[ (\lambda, m_i); (\beta_i, \varphi_i); (\gamma, \psi_i); x_1, \ldots, x_r \left[ \frac{1}{2} \left( 1 + (1 + 4t)^{1/2} \right) \right]^m_i \right].
\]
\[
(3.10)
\]
Putting $a = \lambda - 1$ and
\[
f(x) \equiv F^{(\vartheta)} \left[ (\lambda, m_i); (\beta_i, \varphi_i); (\gamma, \psi_i); \right.
\]
\[
\left. x_1 \left( \frac{2 - m t^{2-m}}{(1-m)^{1-m} t} \right)^{m_i}, \ldots, x_r \left( \frac{2 - m t^{2-m}}{(1-m)^{1-m} t} \right)^{m_i} \right].
\]
in (2.1) and operating $T_k$ on $t$ alone, we get the general result
\[
\sum_{n=0}^{\infty} \frac{(-1)^n n^2 n^2 (\lambda)_{mn}}{n! (\lambda)_{(m-1)n}}
\]
\[
\cdot \sum_{k_1, \ldots, k_r=0}^{\infty} \frac{\lambda + mn}{1-m} \left( \frac{\lambda + mn - m}{1-m} \right)^{x_1^{k_1} \cdots x_r^{k_r}, \Sigma_{k_i}}
\]
\[
\cdot \frac{\lambda + (m-1)n}{1-m} \left( \frac{\lambda + (m-1)n + 1-m}{2-m} \right)^{\Sigma_{k_i}}.
\]
\[
\sum_{n=0}^{\infty} (-1)^n \frac{(\lambda)_m t^n}{(\lambda)_{(m-1)n} n!}
\left[ (\lambda, m); \left( \frac{\lambda+mn}{1-m}, m \right), \ldots, \left( \frac{\lambda+mn-m}{1-m}, m \right); (\beta, \varphi); x_1 \cdots x_r \right]

= \frac{(1+v)^{\lambda}}{1-(m-1)v} F_s^c \left[ (\lambda, m); (\beta, \varphi); (\gamma, \psi); \right.

\left. x_1 \left\{ \frac{(2-m)^{2-m}(1+v)}{(1-m)^{1-m}} \right\}^m, \ldots, x_r \left\{ \frac{(2-m)^{2-m}(1+v)}{(1-m)^{1-m}} \right\}^m \right],
\]

(3.11)

where \( v = tz(1+v)^m \). Putting \( z = 1 \) in (3.11) we get

\[
\frac{1}{\lambda} \frac{(\lambda)_{m' t'^n}}{(\lambda)_{(m'-1)n} n!}
\left[ (\lambda, m'); \left( \frac{\lambda+mn}{1-m}, m \right), \ldots, \left( \frac{\lambda+mn-m}{1-m}, m \right); (\beta, \varphi); x_1 \cdots x_r \right]

= \frac{(1+v)^{\lambda}}{1-(m-1)v} F_s^c \left[ (\lambda, m); (\beta, \varphi); (\gamma, \psi); \right.

\left. x_1 \left\{ \frac{(2-m)^{2-m}}{(1-m)^{1-m}} t(1+v) \right\}^m, \ldots, x_r \left\{ \frac{(2-m)^{2-m}}{(1-m)^{1-m}} t(1+v) \right\}^m \right],
\]

(3.12)

where \( v = t(1+v)^m \).

4. Agarwal [8] defined G-function as follows

\[
G_{p,q,r,s,t}^{m_1,m_2,n_1,n_2} = \frac{1}{4\pi^2} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \frac{\varphi(\xi + \eta)\psi(\xi, \eta) \xi y^n d\xi d\eta}{4\pi^2}
\]
where
\[
\psi(\xi, \eta) = \frac{\prod_{j=1}^{m} \Gamma(\beta_j - \xi) \prod_{j=m+1}^{q} \Gamma(\gamma_j + \xi)}{\prod_{j=m+1}^{q} \Gamma(1 - \beta_j + \xi) \prod_{j=1}^{m} \Gamma(1 - \gamma_j - \xi)} \cdot \frac{\prod_{j=1}^{m} \Gamma(\beta_j' - \eta) \prod_{j=m+1}^{q} \Gamma(\gamma_j' + \eta)}{\prod_{j=m+1}^{q} \Gamma(1 - \beta_j' + \eta) \prod_{j=1}^{m} \Gamma(1 - \gamma_j' - \eta)}
\]
\[
\phi(\xi + \eta) = \frac{\prod_{j=1}^{m} \Gamma(1 - \epsilon_j + \xi + \eta)}{\prod_{j=m+1}^{q} \Gamma(\epsilon_j - \xi - \eta) \prod_{j=1}^{m} \Gamma(\epsilon_j + \xi + \eta)}
\]

and \(0 < m_1 < q, 0 < m_2 < q, 0 < v_1 < t, 0 < v_2 < t, 0 < n < p\). Putting \(a = \alpha, m = \beta + 1\) and

\[
f(x) = G_{p_1, p_2, q_1, q_2}^{m_1, m_2, n_1, n_2, m_3, m_4, n_3, n_4}
\]

in (2.1), we get on substituting \(z_1 = x^2, z_2 = x\) and \(z = t/x\) after operation

\[
\sum_{n=0}^{\infty} \frac{(-1)^n t^n}{n!} G_{p_1+1, p_2, q_1+1, q_2}^{m_1+1, m_2, n_1, n_2, m_3, m_4, n_3, n_4}
\]

\[
= \frac{(1 + v)^{1 - \alpha}}{(1 - \beta v)} G_{p_1, p_2, q_1, q_2}^{m_1, m_2, n_1, n_2, m_3, m_4, n_3, n_4}
\]

where \(v = t(1 + v)^{\beta + 1}\), which is due to Sharma and Abiodun [1].

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**References**


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