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SOME REMARKS ON THE SEQUENCES OF EXHAUSTIVE MEASURES AND UNIFORM BOUNDEDNESS OF MEASURES

SURJIT SINGH KHURANA

ABSTRACT. Nikodym-type theorems and uniform boundedness theorems for measures on Boolean algebra with interpolation property are derived from corresponding results for σ-algebras.

In [3], [6], some Vitali-Hahn-Saks and Nikodym-type theorems are proved about the measures on a Boolean algebra $\mathfrak{A}$ with the interpolation property, called the property I. In this note we prove that these results are simple consequences of well-known results.

For a Hausdorff locally convex space E, always assumed to be over K, the field of real or complex numbers, E' will denote its topological dual [4]. A Hausdorff Abelian topological group G is called normed if there exists a function $|\cdot|: G \to [0, \infty)$ satisfying the conditions (i) $|x| = 0 \Leftrightarrow x = 0$, (ii) $|x + y| \leqslant |x| + |y|$, such that the topology of G comes from the metric P, $P(x,y) = |x - y|$ [1]. A Boolean algebra $\mathfrak{A}$ is said to have the property I if for any two sequences $\{x_n\}$, $\{y_m\}$ in $\mathfrak{A}$ with $x_n \uparrow, y_m \downarrow$, and $x_n \leqslant y_m$, $\forall m, n$, there exists an $x \in \mathfrak{A}$ such that $x_n \leqslant x \leqslant y_m$, $\forall m, n$ [5]. We shall use the theory of submeasures developed in [1] (in [1], submeasures are defined on the algebra of subsets of a set; by considering the clopen subsets of the Boolean space of a Boolean algebra, they can be defined on a Boolean algebra).

LEMMA 1. *Let $\mathfrak{A}$ be a Boolean algebra with the property* I, $\mu: \mathfrak{A} \to [0, \infty)$ *a finite submeasure and $\eta = \{x \in \mathfrak{U}, \mu(x) = 0\}$. Then $\mathfrak{A}/\eta$ is a σ-complete Boolean algebra.*

Proof is very straightforward and similar to [5, Lemma 3.3, p. 275], and as such is omitted.

LEMMA 2. *Let 2^N be the set of all subsets of N, the set of natural numbers, and $\nu_n: 2^N \to G$ be a sequence of exhaustive finitely additive set functions such that* $\lim \nu_n(M)$ *exists, $\forall M \in 2^N$. G being a Hausdorff Abelian topological group. Then $\{\nu_n\}$ are uniformly exhaustive.*

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This is proved in [2, Theorem (BJ), p. 726].

THEOREM 3 [3, Theorem 3.1, p. 111]. *Let* $\mathfrak{A}$ *be a Boolean algebra with the property* I, *G a Hausdorff Abelian topological group, and* μ_n: $\mathfrak{A} \to G$ *a sequence of exhaustive* ($\equiv$ *strongly bounded,* [1]) *finitely additive set functions such that* $\lim \mu_n(b)$ *exists for every* $b \in \mathfrak{A}$. *Then* $\{\mu_n\}$ *are uniformly exhaustive.*

PROOF. Since every group is a subgroup of a product of normed groups, we can assume that $(G, |\cdot|)$ is normed. Every μ_n gives a submeasure $\dot{\mu}_n$,

$$\dot{\mu}_n(b) = \sup\{|\mu_n(a)|: a \in \mathfrak{A}, a < b\}$$

which is exhaustive and finite [1, II, Corollary 4.11, p. 279]. The submeasure

$$\mu = \sum_{n=1}^{\infty} \frac{1}{2^n(1 + \dot{\mu}_n(1))} \dot{\mu}_n$$

is finite and exhaustive. By Lemma 1, $\mathfrak{A}_0 = \mathfrak{A}/\eta$ is a σ-complete Boolean algebra with $\eta = \{x \in \mathfrak{A}: \mu(x) = 0\}$. Let $x \to \hat{x}$ be the canonical mapping $\mathfrak{A} \to \mathfrak{A}_0$. $\bar{\mu}_n$: $\mathfrak{A}_0 \to G$, $\bar{\mu}_n(\hat{x}) = \mu_n(x)$ is well defined. Fix a disjoint sequence $\{y_i\}$ in $\mathfrak{A}$ and define ν_n: $2^N \to G$, $\nu_n(M) = \bar{\mu}_n(\bigvee_{i \in M}\hat{y}_i)$. Since ν_n are exhaustive (this follows from the fact that for any disjoint sequence $\{M_j\}$ in 2^N, $\{\bigvee_{i \in M_j}\hat{y}_i\}$ are disjoint), and $\lim \nu_n(M)$ exists, $\forall M \in 2^N$, they are uniformly exhaustive, by Lemma 2. The result now follows.

THEOREM 4 [3, Theorem 2.1, p. 105]. *Let* $\mathfrak{A}$ *be a Boolean algebra with the property* I, *E a Hausdorff locally convex space, and H a set of bounded finitely additive set functions* $\mathfrak{A} \to E$ *such that* $\{\lambda(b): \lambda \in H\}$ *is a bounded subset of E, for every* $b \in \mathfrak{A}$. *Then* $P = \{\lambda(b): \lambda \in H, b \in \mathfrak{A}\}$ *is bounded in E.*

PROOF. Fix $f \in E'$. $f \circ H$ is a set of bounded finitely additive, scalar-valued set functions on $\mathfrak{A}$ and are pointwise bounded on $\mathfrak{A}$. By [5, Theorem 3.2] $\{f \circ \lambda(b): \lambda \in H, b \in \mathfrak{A}\}$ is bounded. This means P is weakly bounded in E and so is bounded in E [4].

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DEPARTMENT OF MATHEMATICS, UNIVERSITY OF IOWA, IOWA CITY, IOWA 52242