MINIMAL PRIMITIVE IDEALS OF GCR C*-ALGEBRAS

PHILIP GREEN

ABSTRACT. We show that a minimal primitive ideal of a GCR algebra may contain the maximal CCR ideal of the algebra, thus giving a negative answer to a question of J. Dixmier.

In [2, Remarque C, p. 111] (see also [3, 4.7.8]), Dixmier has observed that for a GCR C*-algebra A, any primitive ideal not containing the maximal CCR ideal I of A is in fact a minimal primitive ideal, and he asks whether the converse is true. In terms of the Jacobson topology of the primitive ideal space Prim A, one may rephrase his question as follows (since, by [3, 3.2.1, 3.2.2, and 4.7.15], \( \{ P \in \text{Prim} A : P \supseteq I \} \) is the maximal open \( T_1 \) subset of Prim A): If \( P \in \text{Prim} A \) is not contained in the closure of any other point of Prim A, must \( P \) be contained in the maximal open \( T_1 \) subset of Prim A; that is, must \( P \) have an open \( T_1 \) neighborhood?

We provide a negative answer to this question by considering the C*-algebra \( C^*(\mathbb{R}, X) \) associated (in the sense of [4, p. 890]) to a transformation group \((\mathbb{R}, X)\), where \( X \) is a certain closed subset of \( \mathbb{R}^3 \) on which \( \mathbb{R} \) acts. \((\mathbb{R}, X)\) is described below by giving the (countably many) orbits into which \( X \) is partitioned under \( \mathbb{R} \), together with, for each orbit, an \( \mathbb{R} \)-equivariant map of \( \mathbb{R} \) (which acts on itself by \((t, s) \rightarrow t + s\)) onto the orbit; these maps determine the \( \mathbb{R} \)-action completely.

There is one family of "vertical" orbits \( Y_n, n = 0, 1, \ldots \), passing through the points \( y_0 = (0, 0, 0) \) and \( y_n = (2^{-2n}, 0, 0) \) \((n > 1)\) respectively, with the maps \( \mathbb{R} \rightarrow Y_n, t \rightarrow ty_n \) given by

\[
\begin{align*}
ty_0 &= (0, t, 0), \\
y_n &= (2^{-2n}, t, 0), \quad n > 1.
\end{align*}
\]

There is another family of orbits \( Z_m, n = 1, 2, \ldots \), such that \( Y_n \) lies in the closure of \( Z_n \) for each \( n > 1 \). \( Z_n \) passes through \( z_n = (2^{-2n+1}, 0, 0) \) and is described by (here \( m \) ranges over the integers \( > n + 1 \)):
Thus $Z_n$ consists of an initial infinite vertical segment, together with a sequence of finite vertical segments, increasing in length and converging to $Y_n$, which are connected by half circles (each running from the top of one segment to the bottom of the next). Since the half circles increase in diameter as they approach the planes $x = y_n$, $n = 0, 1, \ldots$, they have no point of accumulation, which ensures that $X = (\bigcup_{n=0}^{\infty} Y_n) \cup (\bigcup_{m=1}^{\infty} Z_n)$ is closed (and hence locally compact); and since points on the vertical pieces are all moved by $R$ “at the same rate”, $(R, X)$ is jointly continuous.

By [4, Theorems 2.1 and 2.2] the $C^*$-algebra $A = C^*(R, X)/J$ (where $J$ is the intersection of the kernels of the “induced” representations of $C^*(R, X)$; although we do not need to know it here, $J = (0)$, by [6, Proposition 2.2]) is Type I (and hence GCR, by [5]) and $\text{Prim } A$ is homeomorphic to the orbit space $X/R$. Now $Y_0$ is not contained in the closure of any other point in $X/R$, but every neighborhood of $Y_0$ contains points $Z_n$ and $Y_n \in \{Z_n\}^-$ and so is not $T_1$. So the primitive ideal corresponding to $\{Y_0\}$ is minimal but has no $T_1$ neighborhood of $\text{Prim } A$.

This algebra has CCR composition series of length two. It may be regarded as a simple example of a type of phenomenon, quite different from that considered in [1], which one must consider in classifying GCR algebras via the extensions that arise at each step of their CCR composition series.

References


School of Mathematics, Institute for Advanced Study, Princeton, New Jersey 08540

Current address: Department of Mathematics, Columbia University, New York, New York 10027