PARACOMPACTNESS, METACOMPACTNESS, AND SEMI-OPEN COVERS

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Abstract. Paracompactness and metacompactness are characterized in terms of locally finite and point-finite semi-open refinements of open covers. It follows from one of these characterizations that a continuous image of a paracompact space under a pseudo-open and compact mapping is metacompact.

1. On semi-open covers. For the meaning of concepts used without definition in this paper, see [3]; note, however, that we do not require paracompact spaces or metacompact spaces to satisfy any separation axioms.

Throughout the following, \( X \) denotes a topological space. Let \( \mathcal{E} \) be a cover of \( X \). For each \( x \in X \), we let \( (\mathcal{E})_x = \{ L \in \mathcal{E} | x \in L \} \). Note that we have \( \text{St}(x, \mathcal{E}) = \bigcup (\mathcal{E})_x \) for each \( x \in X \). If the set \( \text{St}(x, \mathcal{E}) \) is a neighborhood of \( x \) for each \( x \in X \), then we say that \( \mathcal{E} \) is a semi-open cover of \( X \). For some properties of semi-open covers, see [6]. When \( \mathcal{N} \) is a cover of \( X \), we say that \( \mathcal{N} \) is an \( F \)-refinement of the cover \( \mathcal{E} \) if each set \( N \in \mathcal{N} \) is contained in some finite union of sets of the family \( \mathcal{E} \).

Lemma 1.1. A locally finite semi-open cover of a topological space has a locally finite closed \( F \)-refinement.

Proof. Let \( \mathcal{E} \) be a locally finite and semi-open cover of \( X \). For each subfamily \( \mathcal{E}' \) of \( \mathcal{E} \), let \( K(\mathcal{E}') = \text{Cl}(\bigcap \mathcal{E}') - \text{Int}(\bigcup (\mathcal{E} \sim \mathcal{E}')) \). Note that if \( \mathcal{E}' \) is infinite, then \( K(\mathcal{E}') = \emptyset \). For each \( \mathcal{E}' \subset \mathcal{E} \), we have \( K(\mathcal{E}') \subset \bigcup \mathcal{E}' \). To see this, let \( x \in K(\mathcal{E}') \). Then \( x \notin \text{Int}(\bigcup (\mathcal{E} \sim \mathcal{E}')) \) and it follows, since \( x \in \text{Int}(\bigcup (\mathcal{E})_x) \), that we have \( (\mathcal{E})_x \cap \mathcal{E}' \neq \emptyset \), in other words, \( x \in \bigcup \mathcal{E}' \).

Since \( x \in K((\mathcal{E})_x) \) for each \( x \in X \), it follows from the foregoing that the closed family \( \mathcal{K} = \{ K(\mathcal{E}') | \mathcal{E}' \subset \mathcal{E} \} \) is an \( F \)-refinement of \( \mathcal{E} \). To show that \( \mathcal{K} \) is locally finite, let \( x \in X \). Since \( \mathcal{E} \) is locally finite, the subfamily \( \mathcal{E}^* = \{ L \in \mathcal{E} | x \in L \} \) is finite and the open set \( O = X - \text{Cl}(\bigcup (\mathcal{E} \sim \mathcal{E}^*)) \) contains \( x \). If \( \mathcal{E}' \subset \mathcal{E} \) and \( K(\mathcal{E}') \cap O \neq \emptyset \), then \( [\text{Cl}(\bigcap \mathcal{E}')] \cap O \neq \emptyset \) and hence \( (\bigcap \mathcal{E}') \cap O \neq \emptyset \). It follows that if \( K(\mathcal{E}') \cap O = \emptyset \), then \( \mathcal{E}' \subset \mathcal{E}^* \); hence the neighborhood \( O \) of \( x \) intersects only finitely many sets of the family \( \mathcal{K} \). □
In the next section, we use the result of Lemma 1.1 to derive a characterization of paracompactness in terms of the existence of locally finite semi-open refinements of certain open covers. It is not difficult to see that a point-finite semi-open (or even open) cover of a topological space does not always have a point-finite semi-open $F$-refinement by closed sets (see the remark following Theorem 2.2 below). To be able to characterize metacompactness in terms of point-finite semi-open refinements, we show that the existence of such refinements implies the existence of certain open refinements.

Let $\mathcal{L}$ and $\mathcal{R}$ be covers of $X$. We say that $\mathcal{R}$ is a point-wise $W$-refinement of $\mathcal{L}$ if for each $x \in X$, there exists a finite subfamily $\mathcal{L}'$ of $\mathcal{L}$ such that for each $N \in (\mathcal{R})_x$, we have $N \subset L$ for some $L \in \mathcal{L}'$.

**Lemma 1.2.** If an open cover of a topological space has a point-finite semi-open refinement, then the cover has an open point-wise $W$-refinement.

**Proof.** Let $\mathcal{L}$ be a point-finite semi-open refinement of an open cover $\mathcal{U}$ of $X$. For each $L \in \mathcal{L}$, let $U(L) \in \mathcal{U}$ be such that $L \subset U(L)$. For each $x \in X$, denote by $\mathcal{U}(x)$ the finite subfamily $\{U(L) | L \in (\mathcal{L})_x\}$ of $\mathcal{U}$ and denote by $V(x)$ the open neighborhood $[\text{Int } St(x, \mathcal{L})] \cap [\cap \mathcal{U}(x)]$ of $x$. We show that the open cover $V = \{V(x) | x \in X\}$ of $X$ is a point-wise $W$-refinement of the cover $\mathcal{U}$. Let $x \in X$ and let $y \in X$ be such that $x \in V(y)$. Then $x \in St(y, \mathcal{L})$ and hence there exists $L \in \mathcal{L}$ such that $x \in L$ and $y \in L$. For the set $U(L)$, we have $U(L) \in \mathcal{U}(x)$ and $V(y) \subset U(L)$. We have shown that for each $V \in (\mathcal{V})_x$, we have $V \subset U$ for some member $U$ of the finite subfamily $\mathcal{U}(x)$ of $\mathcal{U}$. □

Our remaining lemmas deal with the preservation of the property of semi-openness in certain topological operations.

**Lemma 1.3.** Let $\mathcal{L}$ be a point-finite semi-open cover of $X$ and for each $L \in \mathcal{L}$, let $\mathcal{U}(L)$ be a (point-finite) semi-open cover of the subspace $L$ of $X$. Then the family $\mathcal{R} = \bigcup \{\mathcal{U}(L) | L \in \mathcal{L}\}$ is a (point-finite) semi-open cover of $X$.

**Proof.** It is easily seen that the family $\mathcal{R}$ is point-finite if the families $\mathcal{L}$ and $\mathcal{U}(L)$, $L \in \mathcal{L}$, are all point-finite. To show that $\mathcal{R}$ is a semi-open cover of $X$, let $x \in X$. For each $L \in (\mathcal{L})_x$, the set $St(x, \mathcal{U}(L))$ is a neighborhood of $x$ in the subspace $L$ of $X$ and it follows that there exists a neighborhood $O(L)$ of $x$ in $X$ such that $O(L) \cap L = St(x, \mathcal{U}(L))$. The set $O = [St(x, \mathcal{L})] \cap [\cap \{O(L) | L \in (\mathcal{L})_x\}]$ is a neighborhood of $x$ in $X$. We show that $O \subset St(x, \mathcal{R})$. Let $y \in O$. Then $y \in St(x, \mathcal{L})$ and hence there exists $L \in (\mathcal{L})_x$ such that $y \in L$. But then we have $y \in L \cap O(L) = St(x, \mathcal{U}(L)) \subset St(x, \mathcal{R})$. Hence $O \subset St(x, \mathcal{R})$ and the set $St(x, \mathcal{R})$ is a neighborhood of $x$. □

A mapping $f$ from $X$ onto a topological space $Y$ is called pseudo-open ([1]); in [8] these were called $P_1$-mappings) provided that for each $y \in Y$, whenever $U$ is a neighborhood of the set $f^{-1}\{y\}$ in the space $X$, then the set $f(U)$ is a
neighborhood of the point \( y \) in the space \( Y \).

**Lemma 1.4.** Let \( X \) and \( Y \) be topological spaces, let \( \mathcal{L} \) be a semi-open cover of \( X \) and let \( f \) be a pseudo-open mapping from \( X \) onto \( Y \). Then the family \( \mathcal{H} = \{ f(L) | L \in \mathcal{L} \} \) is a semi-open cover of \( Y \).

**Proof.** The conclusion follows directly from the definitions, since we have \( \text{St}(y, \mathcal{H}) = f(\text{St}(f^{-1}(y), \mathcal{L})) \) for every \( y \in Y \).

2. **On paracompactness and metacompactness.** We start by characterizing paracompactness. Recall that a family \( \mathcal{H} \) of sets is **monotone** provided that the relation \( \subset \) of set inclusion is a linear order on \( \mathcal{H} \).

**Theorem 2.1.** A topological space is paracompact if, and only if, every monotone open cover of the space has a locally finite semi-open refinement.

**Proof.** Necessity of the condition is obvious. To prove sufficiency, assume that every monotone open cover of \( X \) has a locally finite semi-open refinement. For every cardinal number \( k \), denote by \( P(k) \) the following proposition: if \( \mathcal{U} \) is an open cover of \( X \) with \( |\mathcal{U}| = k \), then \( \mathcal{U} \) has a locally finite closed \( F \)-refinement. We observe that \( P(k) \) is trivially true for \( k \) finite and we use transfinite induction to show that \( P(k) \) holds in general. Let \( k \) be an infinite cardinal number such that \( P(h) \) holds for every \( h < k \). To show that \( P(k) \) holds, let \( \mathcal{U} \) be an open cover of \( X \) with \( |\mathcal{U}| = k \). We represent \( \mathcal{U} \) in the form \( \mathcal{U} = \{ U_\alpha | \alpha < \gamma \} \), where \( \gamma \) is the initial ordinal ordinal corresponding to the cardinal \( k \). For each \( \alpha < \gamma \), let \( V_\alpha = \bigcup_{\beta < \alpha} U_\beta \). Then the family \( \mathcal{V} = \{ V_\alpha | \alpha < \gamma \} \) is a monotone open cover of \( X \). The cover \( \mathcal{V} \) has a locally finite semi-open refinement and it follows from Lemma 1.1 that \( \mathcal{V} \) has a locally finite closed \( F \)-refinement, say \( \mathcal{K} \). Let \( K \) be a member of the family \( \mathcal{K} \). Then \( K \) is contained in some finite union of sets of the family \( \mathcal{V} \) and it follows, since \( \mathcal{V} \) is a monotone family, that \( K \) is contained in some set of \( \mathcal{V} \). Let \( \alpha(K) < \gamma \) be such that \( K \subset V_{\alpha(K)} \). The family \( \mathcal{W}(K) = \{ X \sim K | \} \cup \{ U_\alpha | \alpha < \alpha(K) \} \) is an open cover of \( X \) and we have \( |\mathcal{W}(K)| < k \). By the induction assumption, \( \mathcal{W}(K) \) has a locally finite closed \( F \)-refinement, say \( \mathcal{S}(K) \). For every \( K \in \mathcal{K} \), the family \( \mathcal{S}(K) = \{ F \cap K | F \in \mathcal{S}(K) \} \) is a locally finite closed cover of the subspace \( K \) of \( X \). Since \( \mathcal{K} \) is a locally finite and closed cover of \( X \), it follows that the family \( \mathcal{S} = \bigcup \{ \mathcal{S}(K) | K \in \mathcal{K} \} \) is also a locally finite and closed cover of \( X \). It is easily seen that every set of the family \( \mathcal{S} \) is contained in some finite union of sets of the cover \( \mathcal{U} \); hence \( \mathcal{S} \) is an \( F \)-refinement of \( \mathcal{U} \). We have shown that \( P(k) \) holds. This completes the induction.

It follows from the foregoing that every open cover of \( X \) has a locally finite closed \( F \)-refinement. Since a directed cover (see [7]) is an \( F \)-refinement of itself, it follows that every directed open cover of \( X \) has a locally finite closed refinement. By Corollary 6 of [7], the space \( X \) is paracompact. □

Theorem 2.1 generalizes some results of J. Mack [7].
It is not known if the existence of point-finite semi-open refinements for all monotone open covers of a topological space is sufficient for the space to be metacompact; however, we have the following result:

**Theorem 2.2.** A topological space is metacompact if, and only if, every open cover of the space has a point-finite semi-open refinement.

**Proof.** Necessity is obvious and sufficiency follows directly from Lemma 1.2 and the result of J. M. Worrell Jr. that a topological space is metacompact if every open cover of the space has an open point-wise $W$-refinement [11].

Using Theorem 2.2 and the technique used in the proof of Theorem 2.1, it can be shown that a topological space is metacompact if every monotone open cover of the space has a point-finite semi-open closed refinement; for normal spaces this condition is also necessary, since every point-finite open cover of a normal space has an open shrinking (see e.g. [3, Theorem 1.5.18]). In general, however, monotone open covers of metacompact spaces do not necessarily have point-finite semi-open closed refinements (to see this, consider the monotone open cover $\{X \sim \{1/k|k > n\}|n \in \mathbb{N}\}$ of the space $X$ of Example 5.3.4 of [3]).

We close this paper with two corollaries to Theorem 2.2.

**Corollary 2.3.** A topological space is metacompact if it has a point-finite semi-open cover such that every set of the cover is contained in some metacompact subspace of the space.

**Proof.** Assume that $X$ has a point-finite semi-open cover $\mathcal{L}$ such that for each $L \in \mathcal{L}$, there exists a metacompact subspace $M(L)$ of $X$ such that $L \subset M(L)$. To show that $X$ is metacompact, let $\mathcal{U}$ be an open cover of $X$. For each $L \in \mathcal{L}$, the family $\mathcal{U}|M(L) = \{U \cap M(L)|U \in \mathcal{U}\}$ is an open cover of the subspace $M(L)$ of $X$ and hence there exists a point-finite open cover $\mathcal{K}(L)$ of the subspace $M(L)$ such that $\mathcal{K}(L)$ is a refinement of $\mathcal{U}|M(L)$. For each $L \in \mathcal{L}$, the family $\mathcal{K}'(L) = \{N \cap L|N \in \mathcal{K}(L)\}$ is a point-finite open cover of the subspace $L$ of $X$. It follows from Lemma 1.3 that the family $\mathcal{K} = \bigcup \{\mathcal{K}'(L)|L \in \mathcal{L}\}$ is a point-finite semi-open cover of $X$ and it is easily seen that the family $\mathcal{K}$ is a refinement of the cover $\mathcal{U}$. By the foregoing and Theorem 2.2, the space $X$ is metacompact.

R. E. Hodel has shown in [5] that the Locally Finite Sum Theorem holds for metacompactness; since a locally finite closed cover is semi-open, Corollary 2.3 generalizes Hodel's result. It is well known that the analogue of the result of Corollary 2.3 for paracompactness is false; for instance, in [4] there is an example of a nonparacompact Moore space that is the union of two open metrizable subspaces. However, the Locally Finite Sum Theorem also holds for paracompactness ([10]; for regular spaces, [9]).

In [2], A. V. Arhangel'skii proved that a continuous image of a metrizable space under a pseudo-open and compact mapping is metacompact and he
asked whether this result remains true if “metrizable” is replaced by “paracompact”; the following result shows that the answer to this question is in the affirmative.

**Corollary 2.4.** A continuous image of a paracompact space under a pseudo-open and compact mapping is metacompact.

**Proof.** Let $X$ be a paracompact space, and let $f$ be a pseudo-open, compact and continuous mapping from $X$ onto a topological space $Y$. To show that $Y$ is metacompact, let $\mathcal{U}$ be an open cover of $Y$. Then the family $\mathcal{V} = \{f^{-1}(U) | U \in \mathcal{U}\}$ is an open cover of $X$. Let $\mathcal{V}$ be a locally finite open refinement of $\mathcal{V}$ and let $\mathcal{M} = \{f(V) | V \in \mathcal{V}\}$. It is easily seen that $\mathcal{M}$ is a refinement of the cover $\mathcal{U}$ of $Y$ and it follows from Lemma 1.4 that $\mathcal{M}$ is a semi-open cover of $Y$. We also see that the family $\mathcal{M}$ is point-finite since for each $y \in Y$, the compact subset $f^{-1}\{y\}$ of $X$ meets only finitely many sets of the locally finite family $\mathcal{V}$. We have shown that every open cover of $Y$ has a point-finite semi-open refinement. By Theorem 2.2, the space $Y$ is metacompact. □

Note that the above proof can be modified so as to yield the following result: a continuous image of a metacompact space under a pseudo-open and finite-to-one mapping is metacompact.

**References**


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