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# CRYSTALLISATIONS OF 2-FOLD BRANCHED COVERINGS OF $S^3$

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ABSTRACT. We describe the construction of a crystallisation of a 2-fold cyclic covering space of $S^3$ branched over a link, from a bridge-presentation of the branch set.

An $n$-dimensional ball-complex is said to be a *contracted triangulation* of its underlying polyhedron if it satisfies the following conditions:

(i) each $n$-ball, considered with all its faces, is abstractly isomorphic to a closed $n$-simplex;

(ii) the number of 0-balls (vertices) is exactly $n + 1$.

A *crystallisation* of a closed, connected PL manifold $M$ of dimension $n$ is the edge-coloured graph, regular of degree $n + 1$, obtained by taking the 1-skeleton of the cellular subdivision dual to a contracted triangulation of $M$, and by labelling the dual of each $(n - 1)$-simplex by the vertex it does not contain. All topological information on $M$ is contained in such an abstract graph.

A contracted triangulation turns out to be a minimal "pseudodissection" (in the sense of [HW]). The advantage of a pseudodissection is that its incidence structure may be simpler (often *much* simpler) than the one of a simplicial complex triangulating the same space, while the cells composing it still are simplexes. When the space is a manifold, minimality yields: (1) the existence of a "minimal" atlas (in the sense of [$\mathbf{P}_1$]), and (2) the representation by a crystallisation, which, as a graph, belongs to a very circumscribed class (see [F]; in dimension 3 the characteristics of this class are very easy to check). For 3-manifolds, crystallisations are not very different from Heegaard diagrams (see [$\mathbf{P}_2$]), with the advantage that the representation is *completely graph-theoretical*, the embedding into a splitting surface being possible but not necessary (also, a crystallisation embeds into *three* generally nonequivalent splitting surfaces). As a consequence, methods for finding invariants,

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which are typical of Heegaard diagrams, can be generalised to any dimension (see **[G]**). Furthermore, moves of crystallisations appear to be simpler and more direct (in any dimension!) than Singer moves in dimension 3. A survey on these items can be found in **[FG$_2$]**.

A general algorithm exists, which generates a contracted triangulation from a (standard) triangulation of any closed, connected *PL* manifold (see **[P$_1$]**, **[FG$_1$]**). Here we give a much faster construction for the case described in the title. For definitions and properties that we use here without quotation, see **[B]** and **[BH]**.

CONSTRUCTION. Given a bridge-presentation of a link $L$, consider the plane graph $\mathcal{P}$ formed by its projection on the plane $z = 0$; $\mathcal{P}$ can always be assumed to be connected.[2] Call $\mathcal{B}_1, \ldots, \mathcal{B}_m$ the projections of the "bridges". We can assume that $L$ intersects all $\mathcal{B}_i$'s at right angles.

(1) Draw, on the plane $z = 0$, $m$ ellipses $\mathcal{E}_1, \ldots, \mathcal{E}_m$ having $\mathcal{B}_1, \ldots, \mathcal{B}_m$, respectively, as major axes, so as to let each of them intersect each arc of $\mathcal{P}$ at most once.[3] Let $\mathcal{V}$ be the set of points of intersection between the ellipses and $L$. $\mathcal{V}$ separates the part of $L$ lying on $z = 0$ into edges; call $\mathcal{C}$ the set of such edges interior to the ellipses, $\mathcal{D}$ the set of edges exterior to them.

Call $\gamma$ the involution on $\mathcal{V}$ which interchanges the end-points of the edges of $\mathcal{C}$, leaving the points of $\bigcup_i(\mathcal{E}_i \cap \mathcal{B}_i)$ fixed; call $\delta$ the involution on $\mathcal{V}$ which interchanges the end-points of the edges of $\mathcal{D}$. $\mathcal{V}$ also separates the ellipses into even numbers of edges; call $\mathcal{F}$ the set of all such edges.

(2) Label all edges of $\mathcal{D}$ with "colour" $\underline{a}$. Then label all edges on $\mathcal{E}_1$ alternatively with $\underline{c}$ and $\underline{d}$, starting arbitrarily. Complete the colouring on $\mathcal{F}$ with $\underline{c}$ and $\underline{d}$, following the rule that each of the "polygons", determined on the plane $z = 0$ by $\mathcal{F} \cup \mathcal{D}$, is to be bounded by edges of only two colours (note that the edges in each boundary different from $\mathcal{E}_1, \ldots, \mathcal{E}_m$, belong alternatively to $\mathcal{F}$ and to $\mathcal{D}$).

(3) Draw a further set $\mathcal{D}'$ of edges, each connecting a pair of points of $\mathcal{V}$ which correspond under the involution $\gamma\delta\gamma$. Label the elements of $\mathcal{D}'$ with colour $\underline{b}$.

The graph $\mathcal{G}$ which has $\mathcal{V}$ as vertex set, and $\mathcal{D} \cup \mathcal{D}' \cup \mathcal{F}$ as edge set, with the above colouring, is regular of degree 4, and no two adjacent edges have the same colour. Figure 1 illustrates the construction for a presentation of the trefoil knot.

Given any edge $\mathfrak{e}$ of $\mathcal{G}$, if $P$, $P'$ denote its end-points, then $\gamma(P)$ and $\gamma(P')$ are end-points of a (unique) edge $\mathfrak{f}$. In fact, if $P$, $P'$ both lie on $\mathcal{E}_i$, $\mathfrak{f}$ is the symmetric of $\mathfrak{e}$ with respect to $\mathcal{B}_i$; if not, $\mathfrak{f}$ is given by step (3) of the construction. Therefore we have:

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[2]This is immediate if $L$ is nonsplitting. If $L$ splits into a number of links, one can isotope arcs of $L$ on the plane $z = 0$, to pass "in and out" under bridges of different components, without changing the link type.

[3]It is not necessary to use ellipses; any drawing continuously deformable to the one described here, works as well.

LEMMA. *$\gamma$ determines a unique involutory automorphism $\vartheta$ of $\mathcal{G}$ which interchanges $\mathcal{D}$ with $\mathcal{D}'$ and $\underline{c}$-coloured edges with $\underline{d}$-coloured edges.* □

We can now prove:

PROPOSITION. *The 4-coloured graph $\mathcal{G}$ is the crystallisation of a closed, connected 3-manifold $M$. Moreover, $M$ is a 2-fold cyclic covering space of $S^3$ branched over $L$.*

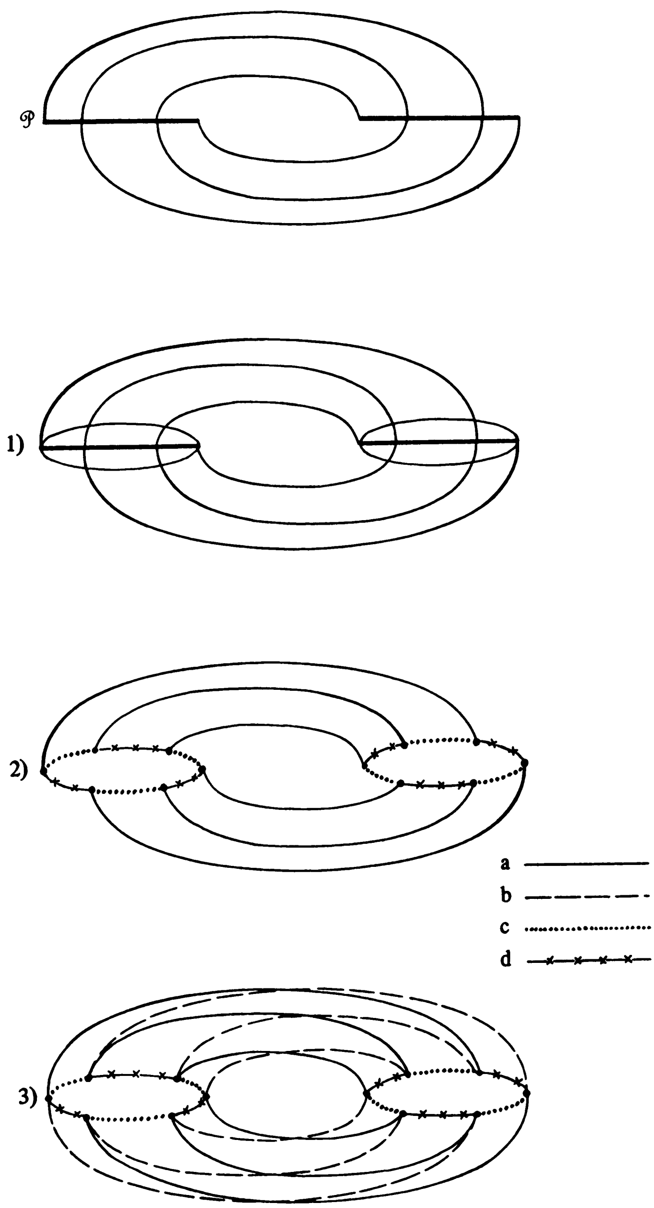


FIGURE 1

PROOF. For each $\underline{x} = \underline{a}, \underline{b}, \underline{c}, \underline{d}$, call $\mathcal{G}_{\underline{x}}$ the partial graph of $\mathcal{G}$ obtained by deleting all $\underline{x}$-coloured edges. In view of Proposition 10 of [$\mathbf{P}_2$], the first part of the statement will be proved, if we show, for each colour $\underline{x}$, that:

($i_{\underline{x}}$) $\mathcal{G}_{\underline{x}}$ is connected;

($ii_{\underline{x}}$) $\mathcal{G}_{\underline{x}}$ can be embedded in a plane so that each 2-cell is bounded by edges with only two colours.

Actually, by the lemma above, we can restrict our attention to $x = \underline{b}, \underline{d}$. ($i_{\underline{b}}$) follows from the assumed connectedness of $\mathcal{P}$. ($ii_{\underline{b}}$) comes from the construction itself (step (2)). To show ($i_{\underline{d}}$), note that $\mathcal{G}_{\underline{d}}$ has the same number of components as the graph $\mathcal{G}'_{\underline{d}}$ obtained by deleting also the edges of $\mathfrak{D}'$ and setting back the ones of $\mathcal{C}$; by shrinking the edges of $\mathcal{C}$ to points, we get the graph $\mathcal{P}$ back, which is connected.

To show ($ii_{\underline{d}}$), note first that to each edge $\mathfrak{e} \in \mathfrak{D}'$ there corresponds a path $\mathfrak{c}\mathfrak{d}\mathfrak{c}'$ connecting the same vertices, with $\mathfrak{c}, \mathfrak{c}' \in \mathcal{C}$, $\mathfrak{d} \in \mathfrak{D}$. So, when drawing (not embedding) $\mathcal{G}$ in the plane $z = 0$, we could have drawn $\mathfrak{e}$ within $\varepsilon$ from the path $\mathfrak{c}\mathfrak{d}\mathfrak{c}'$ and without intersecting it; moreover, we could have chosen that $\mathfrak{e}$ intersects one, of the two ellipses it has to meet, in a $\underline{d}$-coloured edge; then it is bound to intersect also the other ellipse in a $\underline{d}$-coloured edge (by the rule of step (2)). Doing so for each $\mathfrak{e} \in \mathfrak{D}'$, then deleting all the $\underline{d}$-coloured edges, we get an embedding of $\mathcal{G}_{\underline{d}}$ in the plane $z = 0$. For finding out, how the 2-cells are, a comparison with $\mathcal{G}_{\underline{b}}$ turns useful: the $\underline{ac}$-bounded cells of $\mathcal{G}_{\underline{d}}$ are the same as in $\mathcal{G}_{\underline{b}}$; the $\underline{ad}$-bounded cells of $\mathcal{G}_{\underline{b}}$ turn to the $\underline{bc}$-bounded cells of $\mathcal{G}_{\underline{d}}$, when enlarged with regions inside the ellipses, and deprived of the $\varepsilon$-wide strips; the $\underline{cd}$-bounded cells of $\mathcal{G}_{\underline{b}}$ disappear, and the strips build up the $\underline{ab}$-bounded cells of $\mathcal{G}_{\underline{d}}$ (see Figure 2). Thus $\mathcal{G}$ is the crystallisation of a closed, connected 3-manifold $M$.

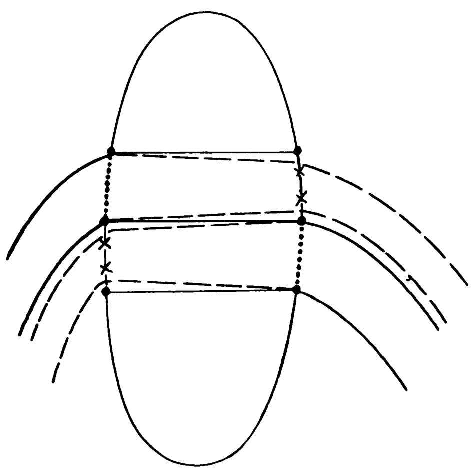

FIGURE 2

$\mathcal{G}$ represents a particular contracted triangulation of $M$; in [$\mathbf{P}_2$] it is shown, how to cut its 3-simplices into prismi, generating a Heegaard splitting $Y$

$\cup_\varphi Y'$ of $M$. There are essentially three ways of doing this, and in each, either handlebody boundary exhibits a copy of $\mathcal{G}$, as 1-skeleton of the decomposition dual to the one induced by the contracted triangulation. The (cellular) identification homeomorphism $\varphi$: $\partial Y \to \partial Y'$ is determined (up to isotopy) by the condition that the two copies of $\mathcal{G}$ are identified by it. In one of these splittings, the $\underline{ab}$-cycles (cycles coloured with $\underline{a}$ and $\underline{b}$) are meridian circles of $Y$, and the $\underline{cd}$-cycles are meridian circles of $Y'$ (the splitting is thus of genus $m - 1$) (see $[\mathbf{P}_2]$).

It is possible to embed $Y$ and $Y'$ in $\mathbf{R}^3$, so that they are invariant under a rotation $T$ of $\pi$ radians about the $x$-axis. As one can see, assuming that all $\mathfrak{B}_i$'s lie on the $x$-axis, it is also possible to embed $\mathcal{G}$ on $\partial Y$, $\partial Y'$ so that $T$ induces the automorphism $\vartheta$ of the lemma on it. This implies that the cellular subdivisions of the handlebodies can be so arranged, that $T$ is cellular on them. Call $D$, $D'$ the orbit spaces $Y/T$, $Y'/T$ respectively (with the induced cellular subdivisions); we will denote by $\pi$ all canonical projections to orbit spaces. Note that the orbit space of each copy of $\mathcal{G}$ under $T$ (i.e. $\mathcal{G}/\vartheta$) is isomorphic to $\mathcal{P}$, by an isomorphism which takes orbits of $\underline{cd}$-cycles to $\mathfrak{B}_i$'s and orbits of $\underline{ab}$-cycles to arcs of $L$.

From what was previously said, $\varphi$ commutes with $T$, hence it induces a homeomorphism $\psi$: $\partial D \to \partial D'$ which identifies the two copies of $\mathcal{P}$ on $\partial D$ and $\partial D'$. The fixed point sets of $T$ in $Y$ and $Y'$ are $Y_x = Y \cap (x\text{-axis})$ and $Y'_x = Y' \cap (x\text{-axis})$ respectively, which are sets of $m$ unlinked, unknotted arcs; these project, by $\pi$, to arcs $\mathfrak{a}_1, \ldots, \mathfrak{a}_m \subset D$, $\mathfrak{a}'_1, \ldots, \mathfrak{a}'_m \subset D'$, with $\mathfrak{a}_i \cap \partial D = \partial \mathfrak{a}_i$, $\mathfrak{a}'_i \cap \partial D' = \partial \mathfrak{a}'_i$. The map

$$\pi: (Y, Y_x) \cup_\varphi (Y', Y'_x) \to \Big(D, \bigcup_i \mathfrak{a}_i\Big) \cup_\psi \Big(D', \bigcup_i \mathfrak{a}'_i\Big),$$

where the second space is a $2m$-plat, is a 2-fold cyclic branched covering projection on $S^3$; the branch set $L'$ is the identification space of $\bigcup_i \mathfrak{a}_i$ and $\bigcup_i \mathfrak{a}'_i$. We can set $D = \{(x, y, z) \in \mathbf{R}^3 | z < 0\} \cup \{\infty\}$, $D' = \{(x, y, z) \in \mathbf{R}^3 | z > 0\} \cup \{\infty\}$, in the one-point compactification of $\mathbf{R}^3$. We can also assume that the two identified copies of $\mathcal{P}$ actually coincide with $\mathcal{P}$, and that $\mathfrak{a}'_1, \ldots, \mathfrak{a}'_m$ are the "bridges" of $L$.

Now, if we substitute $\mathfrak{a}_1, \ldots, \mathfrak{a}_m$ with the orbit spaces of the (meridian) $\underline{ab}$-cycles of $Y$, we get a link ambient isotopic to $L'$ by an isotopy $\mathbf{i}_t$ of $S^3$. But this link is exactly $L$. Composing $\pi$ with $(\mathbf{i}_1)^{-1}$, we get the desired result. □

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