CRYSTALLISATIONS OF 2-FOLD BRANCHED COVERINGS OF $S^3$

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Abstract. We describe the construction of a crystallisation of a 2-fold cyclic covering space of $S^3$ branched over a link, from a bridge-presentation of the branch set.

An $n$-dimensional ball-complex is said to be a contracted triangulation of its underlying polyhedron if it satisfies the following conditions:

(i) each $n$-ball, considered with all its faces, is abstractly isomorphic to a closed $n$-simplex;

(ii) the number of 0-balls (vertices) is exactly $n + 1$.

A crystallisation of a closed, connected PL manifold $M$ of dimension $n$ is the edge-coloured graph, regular of degree $n + 1$, obtained by taking the 1-skeleton of the cellular subdivision dual to a contracted triangulation of $M$, and by labelling the dual of each $(n - 1)$-simplex by the vertex it does not contain. All topological information on $M$ is contained in such an abstract graph.

A contracted triangulation turns out to be a minimal "pseudodissection" (in the sense of [HW]). The advantage of a pseudodissection is that its incidence structure may be simpler (often much simpler) than the one of a simplicial complex triangulating the same space, while the cells composing it still are simplexes. When the space is a manifold, minimality yields: (1) the existence of a "minimal" atlas (in the sense of $[P_1]$), and (2) the representation by a crystallisation, which, as a graph, belongs to a very circumscribed class (see [F]; in dimension 3 the characteristics of this class are very easy to check). For 3-manifolds, crystallisations are not very different from Heegaard diagrams (see $[P_2]$), with the advantage that the representation is completely graph-theoretical, the embedding into a splitting surface being possible but not necessary (also, a crystallisation embeds into three generally nonequivalent splitting surfaces). As a consequence, methods for finding invariants,
which are typical of Heegaard diagrams, can be generalised to any dimension (see [G]). Furthermore, moves of crystallisations appear to be simpler and more direct (in any dimension!) than Singer moves in dimension 3. A survey on these items can be found in [FGj].

A general algorithm exists, which generates a contracted triangulation from a (standard) triangulation of any closed, connected PL manifold (see [P], [FG]). Here we give a much faster construction for the case described in the title. For definitions and properties that we use here without quotation, see [B] and [BH].

Construction. Given a bridge-presentation of a link L, consider the plane graph \( \mathcal{P} \) formed by its projection on the plane \( z = 0 \); \( \mathcal{P} \) can always be assumed to be connected.\(^2\) Call \( \mathcal{B}_1, \ldots, \mathcal{B}_m \) the projections of the “bridges”. We can assume that L intersects all \( \mathcal{B}_i \)'s at right angles.

1. Draw, on the plane \( z = 0 \), \( m \) ellipses \( \mathcal{E}_1, \ldots, \mathcal{E}_m \), having \( \mathcal{B}_1, \ldots, \mathcal{B}_m \), respectively, as major axes, so as to let each of them intersect each arc of \( \mathcal{P} \) at most once.\(^3\) Let \( \mathcal{V} \) be the set of points of intersection between the ellipses and L. \( \mathcal{V} \) separates the part of L lying on \( z = 0 \) into edges; call \( \mathcal{C} \) the set of such edges interior to the ellipses, \( \mathcal{D} \) the set of edges exterior to them.

2. Call \( \mathcal{C} \) the involution on \( \mathcal{V} \) which interchanges the end-points of the edges of \( \mathcal{C} \), leaving the points of \( \bigcup_i (\mathcal{E}_i \cap \mathcal{B}_i) \) fixed; call \( \mathcal{S} \) the involution on \( \mathcal{V} \) which interchanges the end-points of the edges of \( \mathcal{D} \). \( \mathcal{V} \) also separates the ellipses into even numbers of edges; call \( \mathcal{F} \) the set of all such edges.

3. Label all edges of \( \mathcal{D} \) with “colour” a. Then label all edges on \( \mathcal{E}_1 \) alternatively with c and d, starting arbitrarily. Complete the colouring on \( \mathcal{F} \) with c and d, following the rule that each of the “polygons”, determined on the plane \( z = 0 \) by \( \mathcal{F} \cup \mathcal{C} \), is to be bounded by edges of only two colours (note that the edges in each boundary different from \( \mathcal{E}_1, \ldots, \mathcal{E}_m \), belong alternatively to \( \mathcal{F} \) and to \( \mathcal{D} \)).

4. Draw a further set \( \mathcal{D}' \) of edges, each connecting a pair of points of \( \mathcal{V} \) which correspond under the involution \( \mathcal{S} \circ \mathcal{Y} \). Label the elements of \( \mathcal{D}' \) with colour b.

The graph \( \mathcal{G} \) which has \( \mathcal{V} \) as vertex set, and \( \mathcal{D} \cup \mathcal{D}' \cup \mathcal{F} \) as edge set, with the above colouring, is regular of degree 4, and no two adjacent edges have the same colour. Figure 1 illustrates the construction for a presentation of the trefoil knot.

Given any edge \( e \) of \( \mathcal{G} \), if \( P, P' \) denote its end-points, then \( \gamma(P) \) and \( \gamma(P') \) are end-points of a (unique) edge \( \mathcal{S} \). In fact, if \( P, P' \) both lie on \( \mathcal{E}_i \), \( \mathcal{S} \) is the symmetric of \( e \) with respect to \( \mathcal{B}_i \); if not, \( \mathcal{S} \) is given by step (3) of the construction. Therefore we have:

\(^2\)This is immediate if L is nonsplitting. If L splits into a number of links, one can isotope arcs of L on the plane \( z = 0 \), to pass “in and out” under bridges of different components, without changing the link type.

\(^3\)It is not necessary to use ellipses; any drawing continuously deformable to the one described here, works as well.
**Lemma.** $\gamma$ determines a unique involutory automorphism $\sigma$ of $\mathcal{G}$ which interchanges $\mathcal{D}$ with $\mathcal{D}'$ and $c$-coloured edges with $d$-coloured edges. □

We can now prove:

**Proposition.** The 4-coloured graph $\mathcal{G}$ is the crystallisation of a closed, connected 3-manifold $M$. Moreover, $M$ is a 2-fold cyclic covering space of $S^3$ branched over $L$.
Proof. For each $x = a, b, c, d$, call $\mathcal{G}_x$ the partial graph of $\mathcal{G}$ obtained by deleting all $x$-coloured edges. In view of Proposition 10 of [P2], the first part of the statement will be proved, if we show, for each colour $x$, that:

(i) $\mathcal{G}_x$ is connected;

(ii) $\mathcal{G}_x$ can be embedded in a plane so that each 2-cell is bounded by edges with only two colours.

Actually, by the lemma above, we can restrict our attention to $x = b, d$. (i) follows from the assumed connectedness of $\mathcal{G}$. (ii) comes from the construction itself (step (2)). To show (ii), note that $\mathcal{G}_d$ has the same number of components as the graph $\mathcal{G}_b'$ obtained by deleting also the edges of $\mathcal{G}'$ and setting back the ones of $\mathcal{G}$; by shrinking the edges of $\mathcal{G}$ to points, we get the graph $\mathcal{G}$ back, which is connected.

To show (ii), note first that to each edge $e \in \mathcal{G}'$ there corresponds a path $cbe'$ connecting the same vertices, with $c, c' \in \mathcal{G}, b \in \mathcal{G}$. So, when drawing (not embedding) $\mathcal{G}$ in the plane $z = 0$, we could have drawn $e$ within $e$ from the path $cbe'$ and without intersecting it; moreover, we could have chosen that $e$ intersects one, of the two ellipses it has to meet, in a $d$-coloured edge; then it is bound to intersect also the other ellipse in a $d$-coloured edge (by the rule of step (2)). Doing so for each $e \in \mathcal{G}'$, then deleting all the $d$-coloured edges, we get an embedding of $\mathcal{G}_d$ in the plane $z = 0$. For finding out, how the 2-cells are, a comparison with $\mathcal{G}_b$ turns useful: the $ac$-bounded cells of $\mathcal{G}_d$ are the same as in $\mathcal{G}_b$; the $ad$-bounded cells of $\mathcal{G}_b$ turn to the $bc$-bounded cells of $\mathcal{G}_d'$, when enlarged with regions inside the ellipses, and deprived of the $e$-wide strips; the $cd$-bounded cells of $\mathcal{G}_b$ disappear, and the strips build up the $ab$-bounded cells of $\mathcal{G}_d'$ (see Figure 2). Thus $\mathcal{G}$ is the crystallisation of a closed, connected 3-manifold $M$.

Figure 2

$\mathcal{G}$ represents a particular contracted triangulation of $M$; in [P2] it is shown, how to cut its 3-simplices into prismi, generating a Heegaard splitting $Y$.
There are essentially three ways of doing this, and in each, either handlebody boundary exhibits a copy of $\mathcal{G}$, as 1-skeleton of the decomposition dual to the one induced by the contracted triangulation. The (cellular) identification homeomorphism $\varphi$: $\partial Y \to \partial Y'$ is determined (up to isotopy) by the condition that the two copies of $\mathcal{G}$ are identified by it. In one of these splittings, the $ab$-cycles (cycles coloured with $a$ and $b$) are meridian circles of $Y$, and the $cd$-cycles are meridian circles of $Y'$ (the splitting is thus of genus $m - 1$) (see [P2]).

It is possible to embed $Y$ and $Y'$ in $\mathbb{R}^3$, so that they are invariant under a rotation $T$ of $\pi$ radians about the $x$-axis. As one can see, assuming that all $\mathcal{B}_i$'s lie on the $x$-axis, it is also possible to embed $\mathcal{G}$ on $\partial Y$, $\partial Y'$ so that $T$ induces the automorphism $\theta$ of the lemma on it. This implies that the cellular subdivisions of the handlebodies can be so arranged, that $T$ is cellular on them. Call $D$, $D'$ the orbit spaces $Y / T$, $Y' / T$ respectively (with the induced cellular subdivisions); we will denote by $\pi$ all canonical projections to orbit spaces. Note that the orbit space of each copy of $\mathcal{G}$ under $T$ (i.e. $\mathcal{G} / \theta$) is isomorphic to $\mathcal{P}$, by an isomorphism which takes orbits of $\mathcal{G}$ to $\mathcal{P}$, and orbits of $ab$-cycles to arcs of $L$.

From what was previously said, $\varphi$ commutes with $T$, hence it induces a homeomorphism $\psi$: $\partial D \to \partial D'$ which identifies the two copies of $\mathcal{P}$ on $\partial D$ and $\partial D'$. The fixed point sets of $T$ in $Y$ and $Y'$ are $Y_x = Y \cap (x$-axis) and $Y'_x = Y' \cap (x$-axis) respectively, which are sets of $m$ unlinked, unknotted arcs; these project, by $\pi$, to arcs $a_1, \ldots, a_m \subset D$, $a'_1, \ldots, a'_m \subset D'$, with $a_i \cap \partial D = \partial a_i$, $a'_i \cap \partial D' = \partial a'_i$. The map

$$\pi: (Y, Y_x) \cup_{\varphi} (Y', Y'_x) \to \left( D, \bigcup_i a_i \right) \cup_{\psi} \left( D', \bigcup_i a'_i \right).$$

where the second space is a $2m$-plat, is a 2-fold cyclic branched covering projection on $S^3$; the branch set $L'$ is the identification space of $\bigcup a_i$ and $\bigcup a'_i$. We can set $D = \{(x, y, z) \in \mathbb{R}^3 | z < 0\} \cup \{\infty\}$, $D' = \{(x, y, z) \in \mathbb{R}^3 | z > 0\} \cup \{\infty\}$, in the one-point compactification of $\mathbb{R}^3$. We can also assume that the two identified copies of $\mathcal{P}$ actually coincide with $\mathcal{P}$, and that $a'_1, \ldots, a'_m$ are the “bridges” of $L$.

Now, if we substitute $a_1, \ldots, a_m$ with the orbit spaces of the (meridian) $ab$-cycles of $Y$, we get a link ambient isotopic to $L'$ by an isotopy $i$, of $S^3$. But this link is exactly $L$. Composing $\pi$ with $(i_i)^{-1}$, we get the desired result. □

References


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