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SHORTER NOTES

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A SHORT PROOF OF A VERSION OF ASPLUND'S NORM AVERAGING THEOREM

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ABSTRACT. A short proof is given of a somewhat weaker version of Asplund's result on averaging smooth and rotund norms in Banach spaces.

In 1967 E. Asplund [1] found a general construction, which, in the case of locally uniformly rotund (LUR) norms, gives

THEOREM 1 (ASPLUND). *If a Banach space X admits an equivalent LUR norm $\|\cdot\|_1$ and an equivalent norm $\|\cdot\|_2$ whose dual norm is LUR, then X admits an equivalent LUR norm $|||\cdot|||$ whose dual norm is also LUR.*

Recall that an LUR norm is one which satisfies $\lim_j\|x_j - x\| = 0$ whenever $x_j, x \in X$ and $\lim_j 2(\|x_j\|^2 + \|x\|^2) - \|x + x_j\|^2 = 0$.

We give here a short proof of the following weaker version of Theorem 1:

THEOREM 1′ (ASPLUND). *Under the same assumptions as in Theorem 1, X admits an equivalent norm $|||\cdot|||$ which is LUR and Fréchet differentiable (on $X \setminus \{0\}$).*

PROOF OF THEOREM 1′. For $n \geqslant 3$ let $\|f\|_n^* = (\|f\|_1^{*2} + n^{-1}\|f\|_2^{*2})^{1/2}$. Each $\|\cdot\|_n^*$ is clearly an LUR equivalent norm on X^*, dual to some norm $\|\cdot\|_n$ on X. Furthermore, $\lim_n \|x\|_n = \|x\|_1$ uniformly on bounded sets of X. Since each $\|\cdot\|_n^*$ is LUR, the norm $\|\cdot\|_n$ is Fréchet differentiable (cf. e.g. [2]). Consider the norm $|||x||| = (\sum_{n=3}^{\infty}2^{-n}\|x\|_n^2)^{1/2}$; this is an equivalent norm on X. Since the differentials $(\|\cdot\|_n^2)'$ of $\|\cdot\|_n^2$ are uniformly bounded on bounded sets of X, the norm $|||\cdot|||$ is Gâteaux differentiable and the differential $(|||\cdot|||^2)'$ is norm-norm continuous (as all $(\|\cdot\|_n^2)'$ are such–see e.g. [2]). Thus $|||\cdot|||$ is Fréchet differentiable. To see that $|||\cdot|||$ is LUR, suppose x_j, $x \in X$, and $\lim_j 2(|||x_j|||^2 + |||x|||^2) - |||x_j + x|||^2 = 0$. Then the same is true for any $\|\cdot\|_n$ and since $\{x_j\}$ is then necessarily bounded and $\lim_n \|x\|_n =$

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$\|x\|_1$ uniformly on bounded sets, we have $\lim_j 2(\|x_j\|_1^2 + \|x\|_1^2) - \|x_j + x\|_1^2 = 0$. So, by LUR of the norm $\|\cdot\|_1$, we have $\lim_j \|x_j - x\|_1 = 0$.

REMARK. The above argument also works for other properties (like rotundity, uniform rotundity, etc.). In the case where there is exact duality between a differentiability and a rotundity notion (e.g. uniform rotundity and uniform Fréchet differentiability, or rotundity and Gâteaux differentiability in reflexive spaces), our proof gives the original Theorem 1.

REFERENCES

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