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A GENERALIZATION OF A THEOREM OF S. PICCARD

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The following theorem is due to S. Piccard [**2**, p. 30]:

"The difference of two second category Baire sets (see [**1**]) contains a nonempty open set".

For various generalizations of this result the reader is referred to [**3**] and [**4**], where he can also find some more references.

In this note we give a short proof of a generalization of Piccard's theorem.

Let $f\colon X \times X \to X$. We define $f_x\colon X \to X$ and $f^y\colon X \to X$ by $f_x(y) = f(x,y)$ and $f^y(x) = f(x,y)$ for all x, $y \in X$. Then f is globally solvable, if f is continuous and if there exist two continuous functions ψ, ϕ: $X \times X \to X$ such that $f(x,y) = z$ is equivalent to $x = \psi(y,z)$ and $y = \phi(x,z)$ for all $x, y, z \in X$. It follows that f_x, f^y, ψ^z, ϕ^z are homeomorphisms.

If X is a topological group and $f(x,y) = x \cdot y$, then $\psi(y,z)$ and $\phi(x,z)$ may be taken to be $z \cdot y^{-1}$ and $x^{-1} \cdot z$.

THEOREM (CF. [**4**, SATZ 7]). *Let X be a topological space and let $f\colon X \times X \to X$ be a globally solvable function. If $A, B \subset X$ are of second category and A has the property of Baire, then $f(A \times B)$ contains a nonempty open set and X is a Baire space.*

PROOF. By hypothesis there exists a nonempty open set G such that $G - A$ is of first category. For any set $C \subset X$ let $D(C)$ denote the set of all points of X where C is of second category. Then $D(C) \neq \varnothing$ if and only if C is of second category, $G \cap D(C) = G \cap D(G \cap C)$ and $D(G \cap C) = D(G \cap A \cap C)$ [**1**, pp. 83–85]. Thus

$$f(G \times D(B)) = \bigcup \{f^y G : y \in D(B)\}$$

is a nonempty open set. Since $D\gamma = \gamma D$ for each homeomorphism γ of X, $\varnothing \neq f(G \times D(B)) = \{z \in X\colon\ G \cap \psi^z D(B) \neq \varnothing\} = \{z \in X\colon\ G \cap D(\psi^z G) \neq \varnothing\} = \{z \in X\colon\ G \cap D(G \cap \psi^z B) \neq \varnothing\} = \{z \in X\colon\ G \cap D(G \cap A \cap \psi^z B) \neq \varnothing\} \subset \{z \in X\colon A \cap \psi^z B \neq \varnothing\} = f(A \times B)$.

The global solvability of f implies that X is a homogeneous space. Since $D(X)$ is invariant under every homeomorphism of X, it follows that $D(X) = X$, that is, X is a Baire space.

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