A FIXED POINT THEOREM FOR IMAGE-INTERSECTING MAPPINGS

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Abstract. Many of the known fixed point theorems of order reversing mappings of partially or simply ordered sets into themselves pertain to dense such ordered sets. In this paper a fixed point theorem is given for an order reversing mapping from a not necessarily dense simply ordered set into itself.

Let x and y be elements of a simply ordered set (S, <). By the “open at x and closed at y interval” of S, we mean, as usual, (x, y] or [y, x) depending on whether x < y or y < x.

Based on the above, we prove:

Theorem. Let (S, <) be a nonempty simply ordered set in which every nonempty bounded above subset has a least upper bound. Let f be a mapping from S into S such that f is order reversing, i.e.,

\[ x < y \text{ implies } f(x) > f(y) \text{ for every } x, y \in S \] (1)

and such that every nonempty open at x and closed at f(x) interval H of S has a nonempty intersection with its image F[H], i.e.,

\[ f(H) \cap H \neq \emptyset \text{ where } H = (x, f(x)] \text{ or } H = [f(x), x). \] (2)

Then f has a fixed point.

Proof. Since S is nonempty, a \( \in S \) for some a. If \( a < f(a) \) then by (1) we have \( f(a) > f(f(a)) \), and if \( a > f(a) \) then, again by (1) we have \( f(a) < f(f(a)) \). Thus, there exist elements p and q of S such that:

\[ p < q \text{ with } p < f(p) \text{ and } q = f(p) \] (3)

(Indeed, if \( a < f(a) \) we take \( p = a \) and \( q = f(a) \), and, if \( a > f(a) \) we take \( p = f(a) \) and \( q = f(f(a)) \)).

From (3) it follows that the subset E of S given by:

\[ E = \{ x | p < x < q \text{ and } x < f(x) \} \] (4)
is nonempty and bounded above. Let

$$e = \text{lub } E.$$  \hfill (5)

Clearly,

$$p < e.$$  \hfill (6)

We claim that \(e\) is a fixed point of \(f\), i.e.,

$$e = f(e).$$  \hfill (7)

Let us assume on the contrary. Thus, one of the two cases below must occur:

**First case.** \(e < f(e)\). But then from (6) by (1) and (3) it follows:

$$p < e < f(e) < q.$$  \hfill (8)

Since \(e < f(e)\), by (2) there exists \(m \in S\) such that

$$e < m < f(e) \quad \text{and} \quad e < f(m) < f(e)$$  \hfill (9)

which by (8) implies:

$$p < m < q \quad \text{and} \quad p < f(m) < q.$$  \hfill (10)

However, if \(m < f(m)\) then by (10) and (4) we would have \(m \in E\) which by (5) would imply \(m < e\), contradicting (9). On the other hand, if \(m > f(m)\) then by (1) we would have \(f(m) < f(f(m))\) which by (10) and (4) would imply \(f(m) \in E\), which, in turn, by (5) would imply \(f(m) < e\), again contradicting (9).

Hence, the first case cannot occur.

**Second case.** \(e > f(e)\). Hence, again by (2) there exists \(m \in S\) such that:

$$f(e) < m < e \quad \text{and} \quad f(e) < f(m) < e.$$  \hfill (11)

However, since \(e > f(e)\) and since by (11) we have \(m < e\) and \(f(m) < e\), from (4) and (5) it would then follow that there exists \(h \in S\) such that:

$$m < h \quad \text{and} \quad f(m) < h < f(h)$$

contradicting (1).

Hence, the second case also cannot occur.

Thus, our assumption is false and (7) is established.

**Remark 1.** The following example shows that condition (2) by itself does not imply the existence of a fixed point.

Let \(f\) map the real closed interval \([0, 3]\) into itself (where reals are taken in their usual order) such that \(f(x) = 2.5\) for \(0 < x < 1\) and \(f(x) = 0.5\) for \(1 < x < 2\) and \(f(x) = 1.5\) for \(2 < x < 3\).

It can be readily verified that \(f\) satisfies (2) and yet \(f\) has no fixed point.

**Remark 2.** As mentioned earlier, it is not required that \((S, \leq)\) be dense. Indeed, let \(S = \{a, b\}\) with \(a < b\) and \(f(a) = f(b) = b\). Clearly, the hypotheses of the Theorem are satisfied and yet \(S\) is not dense.

**Remark 3.** The statement of the Theorem remains valid if in it "nonempty open at \(x\) and closed at \(f(x)\) interval" is replaced by "nonempty open at \(f(x)\)
and closed at \( x \) interval” and the proof remains unchanged.

**Remark 4.** The statement of the Theorem does not remain valid if in it “nonempty open at \( x \) and closed at \( f(x) \) interval” is replaced by “nonempty open at \( x \) and open at \( f(x) \) interval”. For instance, let \( S = \{a, b\} \) with \( a < b \) and let \( f(a) = b \) and \( f(b) = a \). Then (since there are no nonempty open intervals in \( S \)) every nonempty open interval of \( S \) has a nonempty intersection with its image under \( f \). However, \( f \) has no fixed point.

Similarly, the same example shows that the statement of the Theorem does not remain valid if in it “nonempty open at \( x \) and closed at \( f(x) \) interval” is replaced by “nonempty closed at \( x \) and closed at \( f(x) \) interval”.

For related ideas see [1] to [8].

The author thanks the referee for valuable suggestions.

**References**


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