

TWO CHARACTERIZATIONS OF POWER COMPACT OPERATORS

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ABSTRACT. It is shown that if T is an operator on a Banach space with the property that for every bounded sequence $\{x_n\}$ there is an integer k such that $\{T^k(x_n)\}$ has a convergent subsequence then T is power compact. The equivalent nonstandard characterization is that for each finite point p in an \aleph_1 -saturated enlargement of the space there is an integer k such that $T^k p$ is near-standard. Similar results are shown to hold for countable families of operators and for operators possessing weakly compact powers.

Let $\{T_n: n = 1, 2, 3, \dots\}$ be a countable set of bounded linear transformations mapping from the Banach space X into the Banach space Y . The main purpose of this note is to show that if, for every bounded sequence $\{x_n\}$ of X there is a positive integer k such that $\{T_k(x_n)\}$ has a convergent (respectively weakly convergent) subsequence in Y , then one of the T_k is compact (respectively weakly compact). Thus if $X = Y$ and $T_k = T^k$ for some fixed operator T we obtain the first result of the abstract; recall that T is *power compact* (or *quasi-compact*) if there is a positive integer k such that T^k is compact. In a Hilbert space setting these operators correspond to the nilpotent elements of the Calkin algebra and are known then to be compact perturbations of nilpotent operators [3]. They also arise naturally as weakly compact operators on such classical spaces as $C(S)$ and $L_1(S, \Sigma, \mu)$ [1, Chapter VI] and have been extensively studied because, for example, they share many of the spectral properties of compact operators [1, p. 579]. In fact it was the observation that the proofs of these spectral results also held for operators possessing the above (apparently) more general compactness property that led the author to examine the question whether the two classes coincided. We show that they do by providing what seems to be a natural proof which extends to give analogous results for operators with weakly compact powers. This nonstandard proof is based on an application of the Baire category theorem to the nonstandard hull of X and Robinson's characterization of compactness [4, p. 93].

We suppose our spaces are embedded in an appropriately large structure

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\mathfrak{N} , and we develop the nonstandard theory in an \mathfrak{N}_1 -saturated enlargement ${}^*\mathfrak{N}$ [2]. The nonstandard hull \hat{X} of X is the quotient space $\text{fin } {}^*X/\mu(0)$ where $\text{fin } {}^*X$ denotes the set of norm-finite elements of *X and $\mu(0)$, the monad of 0, is the set of norm-infinitesimal elements of *X . The hull becomes a normed linear space when given the norm $\|\hat{p}\| = \text{st}(\|p\|)$. Here \hat{p} denotes the equivalence class determined by $p \in \text{fin } {}^*X$ in \hat{X} and st denotes the standard part operator. By a result of Luxemburg [2] \hat{X} is complete under the assumption that ${}^*\mathfrak{N}$ is \mathfrak{N}_1 -saturated. Finally we say $p \in {}^*X$ is *weakly near-standard* if it is near-standard [4, p. 93] when X is given the weak topology.

THEOREM 1. *Suppose the family $\{T_k: k = 1, 2, 3, \dots\}$ has the property that for each $p \in \text{fin } {}^*X$ there is a positive integer k such that $T_k(p)$ is near-standard (respectively weakly near-standard). Then there is a positive integer k such that $T_k(p)$ is near-standard (respectively weakly near-standard) for each $p \in \text{fin } {}^*X$.*

PROOF. For $k = 1, 2, 3, \dots$ we define X_k to be $\{\hat{p}: T_k p \text{ is near-standard (respectively weakly near-standard)}\}$. These are meaningful definitions and as we shall see later each X_k is then a closed subspace of \hat{X} . By the initial assumptions $\bigcup_{k=1}^{\infty} X_k = \hat{X}$ whence, by the Baire category theorem, $X_k = \hat{X}$ for some integer k . This implies $T_k p$ is near-standard (respectively weakly near-standard) for each $p \in \text{fin } {}^*X$. Thus we have the result once we have checked that the X_k are closed. For the first case assume $\hat{p}_n \rightarrow \hat{p}$ where $\text{st}(T_k(p_n)) = y_n$ for each n . Then $\{y_n\}$ is a Cauchy sequence in Y since

$$\|y_l - y_m\| = \text{st}\|T_k(p_l - p_m)\| \leq \|T_k\| \|\hat{p}_l - \hat{p}_m\|,$$

and therefore is convergent as Y is complete. If $y = \lim y_n$ then the inequality

$$\|y - T_k p\| \leq \|y - y_n\| + \|y_n - T_k p_n\| + \|T_k(p_n - p)\|$$

forces the equality $y = \text{st}(T_k p)$. In the other case assume $\hat{p}_n \rightarrow \hat{p}$ where $\text{weak-st}(T_k(p_n)) = y_n$ for each n . This time $\{y_n\}$ is a Cauchy sequence in Y since $\|f(y_l - y_m)\| = \text{st}\|f(T_k(p_l - p_m))\| \leq \|T_k\| \|\hat{p}_l - \hat{p}_m\|$ whenever $f \in S'$, the unit sphere of the dual space Y' . If we let $y = \lim y_n$ then $y = \text{weak-st } T_k(p)$.

COROLLARY 1. *Suppose the family $\{T_k: k = 1, 2, 3, \dots\}$ has the property that for each $p \in \text{fin } {}^*X$ there is a positive integer k such that $T_k(p)$ is near-standard (respectively weakly near-standard). Then there is a positive integer k such that T_k is compact (respectively weakly compact).*

PROOF. Both conclusions are consequences of the theorem and the earlier referred to result of Robinson.

THEOREM 2. *Suppose the family $\{T_k: k = 1, 2, 3, \dots\}$ has the property that for each bounded sequence $\{x_n\}$ of X there is a positive integer k such that*

$\{T_k(x_n)\}$ has a convergent subsequence. Then for each $p \in \text{fin } {}^*X$ there is a positive integer k such that $T_k p$ is near-standard.

PROOF. Since Y is complete the result holds if we show there is an integer k such that for each $\varepsilon > 0$ there exists a point $y \in Y$ such that

$$\|T_k p - y\| < \varepsilon.$$

Let us suppose, to the contrary, that for each $k = 1, 2, 3, \dots$ there exists a constant $\varepsilon_k > 0$ such that for each $y \in Y$

$$\|T_k p - y\| > \varepsilon_k.$$

As a consequence of this we will show that there exists a bounded sequence $\{x_n\}$ in X such that

$$\|T_k(x_i) - T_k(x_j)\| > \varepsilon_k \quad \text{for } k = 1, 2, 3, \dots,$$

provided $\max(i, j) > k$ and $i \neq j$.

Since the existence of this sequence clearly contradicts the assumption of the theorem we are finished once we have established the existence of the sequence. Suppose in fact that we have chosen x_1, x_2, \dots, x_k after having initially chosen x_1 arbitrarily. If A is a standard bound for $\|p\|$ then the (formal) sentence

$$\exists x \in X \|x\| \leq A \ \& \ \|T_i x_j - T_i x\| > \varepsilon_i \quad \text{for } i, j = 1, \dots, k$$

is true in ${}^*\mathfrak{N}$ (p satisfies it there) and so it is true in \mathfrak{N} . This guarantees the existence of a point x_{k+1} in X satisfying

$$\|T_i x_j - T_i x_{k+1}\| > \varepsilon_i \quad \text{for } i, j = 1, \dots, k$$

and thus ultimately that of the sequence.

COROLLARY 2. *Suppose the family $\{T_k: k = 1, 2, 3, \dots\}$ has the property that for each bounded sequence $\{x_n\}$ of X there is a positive integer k such that $\{T_k(x_n)\}$ has a convergent subsequence. Then there is a positive integer k such that T_k is compact.*

PROOF. The result is an immediate consequence of Corollary 1 and Theorem 2.

THEOREM 3. *Suppose the family $\{T_k: k = 1, 2, 3, \dots\}$ has the property that for each bounded sequence $\{x_n\}$ of X there is a positive integer k such that $\{T_k(x_n)\}$ has a weakly convergent subsequence. Then for each $p \in \text{fin } {}^*X$ there is a positive integer k such that $T_k p$ is weakly near-standard.*

PROOF. Clearly, since Y is complete, it is sufficient to show there exists an integer k such that for each $\varepsilon > 0$ there exists a point $y \in Y$ such that for each $f \in S'$,

$$f(T_k p) - f(y) < \varepsilon.$$

In fact this is equivalent [5, Lemma 2] to showing there exists an integer k such that for each $\varepsilon > 0$ there is a finite subset C of Y such that for each

$f \in S'$ there is a point $y \in C$ satisfying

$$f(y - T_k p) < \varepsilon.$$

We prove this last assertion by contradiction. Suppose for $k = 1, 2, 3, \dots$ there exists $\varepsilon_k > 0$ such that for each finite subset C of Y there is an $f \in S'$ such that for each $y \in C$, $f(y - T_k p) > \varepsilon_k$. We claim that as a consequence of this we can construct a bounded sequence $\{x_n\}$ in X and for each integer k a sequence $\{f_n^k: n = k, k + 1, k + 2, \dots\}$ of elements from S' satisfying

$$f_n^k(T_k x_i - T_k x_j) > \varepsilon_k \quad \text{for } 1 < i < n < j.$$

To see this suppose we have chosen x_i for $i = 1, \dots, k$ and f_i^j for $1 < j < i < k$. Then for $l = 1, 2, \dots, k$ we choose functionals f_l^k satisfying

$$f_l^k(T_l(x_i) - T_l(p)) > \varepsilon_l \quad \text{for } i = 1, 2, \dots, k.$$

Arguing as in Theorem 2 it follows that there must be a suitably bounded element x_{k+1} such that

$$f_l^k(T_l(x_i) - T_l(x_{k+1})) > \varepsilon_l \quad \text{for } i, l = 1, 2, \dots, k$$

and this gives the existence of the sequences. The proof is completed once we verify that for each positive integer l the sequence $\{T_l(x_n)\}$ does not have a weakly convergent subsequence. If we suppose that for a given integer k , $\{T_k(x_n)\}$ has a weakly convergent subsequence then by renumbering if necessary we can assume there is an element $y \in Y$ such that

$$T_k(x_j) \rightarrow y \quad \text{weakly as } j \rightarrow \infty.$$

Then

$$f_n^k(T_k x_i - y) \geq \varepsilon_k \quad \text{for } 1 < i < n.$$

But the Banach-Alaoglu theorem ensures that the sequence $\{f_n^k: n = k, k = 1, \dots\}$ has a weak* accumulation point f , and hence, using a diagonal process, there exists a subsequence $\{n_p\}$ such that for each $i = 1, 2, \dots$

$$\lim_{p \rightarrow \infty} f_{n_p}^k(T_k x_i) = f(T^k x_i).$$

It then follows that $f(T_k x_i - y) \geq \varepsilon_k$ for $i = 1, 2, 3, \dots$ contradicting the assumption that $T_k x_i \rightarrow y$ in the weak topology.

COROLLARY 3. *Suppose the family $\{T_k: k = 1, 2, 3, \dots\}$ has the property that for each bounded sequence $\{x_n\}$ of X there is a positive integer k such that $\{T_k(x_n)\}$ has a weakly convergent subsequence. Then there is a positive integer k such that T_k is weakly compact.*

PROOF. This final result is a consequence of Corollary 1 and Theorem 3.

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