THE HAAR FUNCTIONS ALMOST DIAGNOLIZE MULTIPLICATION BY \textit{x}

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Abstract. It is shown that if \(A\) is multiplication by \(x\) on \(L^2(0, 1)\), then the matrix for \(A\) given by the Haar functions has the form diagonal plus Hilbert-Schmidt.

If \(A\) is a self-adjoint operator on a complex separable Hilbert space \(\mathcal{H}\), then \(A\) has the form \(D + K\), where \(D\) is diagonal (i.e., the eigenvectors for \(D\) span \(\mathcal{H}\)) and \(K\) is a Hilbert-Schmidt operator. This fact, which was first proved by von Neumann in [1], has played an important role in several areas of analysis. From the proof, however, there is no reason to expect that the basis that "almost diagonalizes" \(A\) will have a "nice" form. The purpose of this note is to show there is a natural basis (the Haar functions) for \(L^2(0, 1)\) that almost diagonalizes the operator \(A\) defined by

\[(Af)(x) = xf(x), \quad f \in L^2(0, 1).\]

If \(0 < a < b < 1\), write \([a, b]\) for the characteristic function of the indicated interval. The Haar functions are defined as follows:

\[\psi_{00} = [0, 1], \quad \psi_{01} = [0, \frac{1}{2}] - [\frac{1}{2}, 1]\]

and, if \(n \geq 1, 0 < k < 2^n - 1,\)

\[\psi_{nk} = (\sqrt{2})^n (\left[\left[k/2^n, k/2^n + 1/2^{n+1}\right] - \left[\frac{k}{2^n + 1/2^{n+1}}, (k + 1)/2^n\right]\right).\]

It is obvious that the Haar functions form an orthonormal set; and, as their linear span contains the characteristic function of every dyadic subinterval, they form an orthonormal basis.

For each \(n\) and \(k\) write

\[a_{nk} = \|A\psi_{nk}\|^2 - (A\psi_{nk}, \psi_{nk})^2.\]

Easy calculations show \(a_{00} = a_{01} = \frac{1}{12}\) and if \(n \geq 1, 0 < k < 2^n - 1,\) then

\[\|A\psi_{nk}\|^2 = \left(\frac{1}{3}\right)(2^{-2n})(3k^2 + 3k + 1)\] and \( (A\psi_{nk}, \psi_{nk}) = \left(\frac{1}{2}\right)(2^{-n})(2k + 1). \)
Therefore, \( a_{nk} = (\frac{1}{12})(2^{-2n}) \) and
\[
a_{00} + a_{01} + \sum_{n=1}^{\infty} \sum_{k=0}^{2^n-1} a_{nk} = \frac{1}{12} + \frac{1}{12} + \sum_{n=1}^{\infty} (2^n)(2^{-2n})(\frac{1}{12}) = \frac{1}{4}.
\]
If \( K \) denotes the operator whose matrix in the basis \( \{ \phi_{nk} \} \) is 0 on the main diagonal and agrees with the matrix for \( A \) at the off-diagonal entries, then
\[
\|K\|_{HS}^2 = \sum a_{nk} = \frac{1}{4}.
\]
Thus, \( K \) is a Hilbert-Schmidt operator and \( D = A - K \) is diagonal.

References