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$L^1(G)$ AS AN IDEAL IN ITS SECOND DUAL SPACE

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ABSTRACT. Based on general results on Banach modules a short proof of the following criterion due to S. Watanabe [5] is given: A group algebra $L^1(G)$ is a two-sided ideal in its second dual space (equipped with one of the Arens products) if and only if G is compact.

Equipped with one of the Arens products ([1], [2]) the bidual of a group algebra $L^1(G)$ (G a locally compact Hausdorff group) becomes a Banach algebra in which the image of $L^1(G)$ under the canonical embedding is a subalgebra. In [5] and [6] S. Watanabe proved that $L^1(G)$ is a two-sided ideal in its bidual if and only if G is compact.

The sufficiency of the condition had been shown previously by P.-K. Wong [7] using the fact that for compact G, $L^1(G)$ is a dual A^*-algebra.

For the necessity of the condition, Watanabe gave two different proofs: In [5], given a noncompact locally compact group G, functions $x \in L^1(G)$, f, $f_n \in L^\infty(G)$ ($n = 1, 2, \ldots$) are defined, where f is the $\sigma(L^\infty, L^1)$-limit of the sequence f_n. For each n, $x * f_n$ has compact support, while $x * f$ does not even vanish at infinity. From the assumption of $L^1(G)$ being an ideal in its second dual space, one can conclude that $x * f$ would have to be the weak limit of the sequence $x * f_n$, thus contradicting the fact that $x * f$ does not vanish at infinity.

In [6], Watanabe showed in a different way that compactness of G is a necessary condition: Here the first step consists in showing that, given g, $h \in L^1(G)$, the mapping $f \mapsto g * f * h$ is compact on $L^1(G)$ if $L^1(G)$ is a (one-sided) ideal in its bidual. Assuming that G is not compact, a sequence of functions contained in the closure of a set of the form $g * S * g$ ($g \in L^1(G)$, S the unit ball of $L^1(G)$) can be constructed having no cluster point in $L^1(G)$, thus contradicting the relative compactness of the set $g * S * g$. Therefore G must be compact.

In this note we show that this criterion can be proved by purely functional-analytic methods, the property of $L^1(G)$ being a group algebra being of importance only inasmuch as we use the fact that the constant function 1 on G vanishes at infinity if and only if G is compact.

First we have to introduce some notation. If A is a Banach algebra and V,

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W are left (resp. right) Banach A-modules then $H_A(V, W)$ (resp. $H^A(V, W)$) denotes the space of left (right) Banach A-module homomorphisms from V into W. For each Banach space X, let X^* resp. X^{**} denote its dual resp. bidual. We have shown in [3, 4.14, 4.17] that for a left Banach A-module V there exist linear homeomorphisms

$$V^{**} \cong H^A(V^*, A^*) \oplus (V^*A)^{\perp}, \qquad W \cong H_A(A, V) \oplus (V^*A)^{\perp}, \quad (1)$$

provided that A contains a bounded (multiple) right approximate identity. W consists of those elements w of V^{**} for which for all $a \in A$, aw is contained in the canonical image of V in V^{**} (V^* and V^{**} have a natural right resp. left A-module structure!). $(V^*A)^{\perp}$ denotes the annihilator in V^{**} of the essential part V^*A of V^*.

Now we give the announced functional-analytic proof of Watanabe's criterion (actually, we prove a slightly different version, with "two-sided" (ideal) replaced by "right" *or* "left"). Let $A = V = L^1(G)$; then it follows $H_A(A, V) = C_0^*$ and $H^A(V^*, A^*) = C_{\mathrm{ru}}^*$ (see e.g. [4]) where C_{ru} denotes the space of all bounded right uniformly continuous[1] functions on G and C_0 its subspace consisting of the continuous functions on G vanishing at infinity. It is clear that $W = V^{**}$ is equivalent to $C_0 = C_{\mathrm{ru}}$, which, of course, is the case if and only if G is compact. On the other hand, by definition of W, $W = V^{**}$ in the present context means exactly that $L^1(G)$ is a right ideal in its bidual: We have $A = V = L^1(G)$, and hence W is the largest subalgebra of $L^1(G)^{**}$ containing $L^1(G)$ as a right ideal.

The right A-module versions of the formulas (1) show that $L^1(G)$ is a left ideal in its bidual if and only if G is compact.

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[1] "Right uniformly continuous" in the sense of Bourbaki, Hewitt-Ross et al. Following this terminology "C_{lu}" should be replaced by "C_{ru}" in [4] throughout.