SUMMABILITY FACTORS FOR CESÁRO METHODS

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Abstract. It is shown that if each of \( r \) and \( s \) is a nonnegative integer and \( \{j^n\} \) is a complex sequence such that \( \Sigma j^n a_p \) is Cesàro summable of order \( s \) whenever \( \Sigma a_p \) is Cesàro summable of order \( r \), then \( \Sigma j^n f_p a_p \) is Cesàro summable of order \( r \) whenever \( \Sigma a_p \) is Cesàro summable of order \( r \).

In [2, p. 310] Tetsuzo Kojima obtained the following result.

Theorem K. In order for \( \{f_p\} \) to have the property that \( \Sigma f_p a_p \) is Cesàro summable of order \( s \) \( ((C, s)\)-summable) whenever \( \Sigma a_p \) is \( (C, r)\)-summable, it is necessary and sufficient that the following conditions hold:

\[
\sup_n n^{r-s} |f_n| < \infty, \tag{1}
\]

and

\[
\sup_n \frac{1}{A_n^{(s)}(r)} \sum_{i=1}^{n-r-1} A_i^{(r)} \left| \sum_{k=0}^{r+1} \binom{r+1}{k} A_{n-i+1}^{(s-k)} \Delta^{r-(k-1)} f_{i+k} \right| < \infty, \tag{2}
\]

where \( A_n^{(k)} = (n + k - 1)!/(k!(n-1)!) \), \( A_n^{(-k)} = 0 \), \( n, k = 1, 2, 3, \ldots \), and each of \( r \) and \( s \) is a nonnegative integer.

In the present paper, we simplify Kojima's conditions for certain cases, and, as a consequence, are able to see a surprising phenomenon: If \( \Sigma f_p a_p \) is \( (C, s)\)-summable whenever \( \Sigma a_p \) is \( (C, r)\)-summable, then \( \Sigma f_p a_p \) is \( (C, r)\)-summable whenever \( \Sigma a_p \) is \( (C, r)\)-summable.

To simplify Kojima's conditions, we note first that, for \( i \) fixed,

\[
\lim_n \frac{A_{n-i+1}^{(s-k)}}{A_n^{(s)}} = \lim_n \left( \frac{n^{-i+1+s-k-1}}{s-k} \right) = \begin{cases} 
0 & \text{if } k > 0, \\
1 & \text{if } k = 0.
\end{cases} \tag{\ast}
\]

Note that (\ast) holds even if \( s - k < 0 \), since the numerator of the first

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fraction is 0 for this case. Using (*) we find that, for \( i \) fixed,

\[
\lim_n \left[ \frac{A_i^{(r)}}{A_n^{(s)}} \sum_{k=0}^{r+1} \binom{r+1}{k} A_{n-i+1}^{(s-k)} \Delta^{-(k-1)} f_{i+k} \right]
\]

\[
= \sum_{k=0}^{r+1} \left[ \binom{r+1}{k} A_i^{(r)} \Delta^{-(k-1)} f_{i+k} \lim_n \frac{A_{n-i+k}^{(s-k)}}{A_n^{(s)}} \right]
\]

\[
= A_i^{(r)} \Delta^{r+1} f_i
\]

\[
= \left( \frac{i + r - 1}{r} \right) \Delta^{r+1} f_i.
\]

Thus if (2) holds, then

\[
\sum_{i=1}^{\infty} \left( \frac{i + r - 1}{r} \right) |\Delta^{r+1} f_i| < \infty.
\]

(2')

Note the special case of (2') when \( r = 0 \):

\[
\sum_{i=1}^{\infty} |\Delta f_i| < \infty,
\]

(2'')

which means that \( f = \{f_j\} \) is of bounded variation, and consequently is not only bounded but convergent.

Next we show that for \( k < r \) and \( 1 < i < n \),

\[
\frac{A_i^{(r)} A_{n-i+1}^{(s-k)}}{A_n^{(s)}} \leq (s!) (r!) A_i^{(r-k)}.
\]

(**)

We note that (**) holds if \( s < k \), since \( A_{n-i+1}^{(s-k)} = 0 \) for this case. For \( 2 < i < n \) and \( 0 < k < \min\{s, r\} \), we have

\[
\frac{A_i^{(r)} A_{n-i+1}^{(s-k)}}{A_n^{(s)}} = \frac{(i + r - 1)!}{r! (i - 1)!} \frac{(n - i + 1 + s - k - 1)!}{(n - k)! (n - i + 1 - 1)!} \frac{(n + s - 1)!}{s! (n - 1)!}
\]

\[
= \frac{[(i + k - 1) \cdots (i)] [(n - 1) \cdots (n - i + 1)]}{(n + s - 1)(n + s - 2) \cdots (n + s - i - k + 1)}
\]

\[
\cdot \frac{s! (r - k)!}{r! (s - k)!} \cdot \frac{(i + r - 1)!}{(r - k)! (i + k - 1)!}
\]

\[
\leq \frac{(n + k - 1)(n + k - 2) \cdots (n - i + 1)}{(n + s - 1)(n + s - 2) \cdots (n + s - i - k + 1)}
\]

\[
\cdot (s!) (r!) \left( \frac{i + r - 1}{r - k} \right)
\]

\[
\leq (s!) (r!) A_i^{(r-k)}.
\]

We omit similar proofs for the two cases \( k = 0, i > 1 \) and \( k > 0, i = 1 \).

Now suppose (1) and (2') hold and \( s < r \). Then \( f \in \text{BV}_{r+1} \), where \( \text{BV}_j \) is
the set of all bounded complex sequences $x$ such that

$$\sum_{p=1}^{\infty} \left( \frac{p + j - 2 - (j - 1)}{2} \right) |\Delta jx_p| < \infty.$$ 

It is known [1, pp. 350–351] that if $x \in BV_j$, then $x \in BV_t$ for $1 < t < j - 1$. Thus $\sum_{p=1}^{\infty} \left( \frac{p + j - 2 - (j - 1)}{2} \right) |\Delta jx_p| < \infty$ for $1 < t < r + 1$, so that, using $(**)$, we have

$$\frac{1}{A_n^{(s)}} \sum_{i=1}^{n-r-1} A_i^{(r)} \sum_{k=0}^{r} \binom{r+1}{k} A_{n-i+k}^{(r-k)} |\Delta^{r-k} i f_{i+k}|$$

$$< \sum_{i=1}^{n-r-1} \sum_{k=0}^{r} \binom{r+1}{k} (s!) (r!) A_{n-i+k}^{(r-k)} |\Delta^{r-k} i f_{i+k}|$$

$$< (r+1)! (r!) (s!) \sum_{i=1}^{r} \sum_{k=0}^{r} \binom{r+1}{r+k} |\Delta^{r-k} f_{i+k}|$$

and consequently, since the last expression is free of $n$, we see that (2) holds.

Thus we have shown that if (1) and (2') hold and $s < r$, then (2) holds, and we previously showed that (2) implies (2') in any case. Hence we have the following result.

**THEOREM 1.** In order for $f$ to have the property that $\sum f_p a_p$ is $(C, s)$-summable whenever $\sum a_p$ is $(C, r)$-summable, $s < r$, it is necessary and sufficient that (1) and (2') hold.

The following special case of Theorem 1 is especially useful for our purpose:

In order for $f$ to have the property that $\sum f_p a_p$ is $(C, r)$-summable whenever $\sum a_p$ is $(C, r)$-summable, it is necessary and sufficient that $f \in BV_{r+1}$.

**THEOREM 2.** If $\sum f_p a_p$ is $(C, s)$-summable whenever $\sum a_p$ is $(C, r)$-summable, then $\sum f_p a_p$ is $(C, r)$-summable whenever $\sum a_p$ is $(C, r)$-summable.

**PROOF.** The statement is obvious if $s < r$. Suppose $r < s$. From Theorem K and a previous argument, we see that (2') holds. Now since the set of convergent sequences ($(C, 0)$-summable sequences) is a subset of the set of $(C, r)$-summable sequences, we know that $\sum f_p a_p$ is $(C, s)$-summable whenever $\sum a_p$ is $(C, 0)$-summable. Hence from Theorem K and a previous argument, (2'') holds, so that $f$ is bounded. But $f$ bounded and (2') imply that $f \in BV_{r+1}$, and this, by the special case of Theorem 1 mentioned above, means that $\sum f_p a_p$ is $(C, r)$-summable whenever $\sum a_p$ is $(C, r)$-summable.
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