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SUMMABILITY FACTORS FOR CESÀRO METHODS

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ABSTRACT. It is shown that if each of r and s is a nonnegative integer and $\{f_p\}$ is a complex sequence such that $\Sigma f_p a_p$ is Cesàro summable of order s whenever Σa_p is Cesàro summable of order r, then $\Sigma f_p a_p$ is Cesàro summable of order r whenever Σa_p is Cesàro summable of order r.

In [2, p. 310] Tetsuzo Kojima obtained the following result.

THEOREM K. *In order for* $\{f_p\}$ *to have the property that* $\Sigma f_p a_p$ *is Cesàro summable of order* s *((*C, s*)-summable) whenever* Σa_p *is (*C, r*)-summable, it is necessary and sufficient that the following conditions hold*:

$$\sup_n n^{r-s}|f_n| < \infty, \tag{1}$$

and

$$\sup_n \frac{1}{A_n^{(s)}} \sum_{i=1}^{n-r-1} A_i^{(r)} \left| \sum_{k=0}^{r+1} \binom{r+1}{k} A_{n-i+1}^{(s-k)} \Delta^{r-(k-1)} f_{i+k} \right| < \infty, \tag{2}$$

where $A_n^{(k)} = (n+k-1)!/k!(n-1)!$, $A_n^{(-k)} = 0$, $n, k = 1, 2, 3, \ldots,$ *and each of* r *and* s *is a nonnegative integer.*

In the present paper, we simplify Kojima's conditions for certain cases, and, as a consequence, are able to see a surprising phenomenon: If $\Sigma f_p a_p$ is (C, s)-summable whenever Σa_p is (C, r)-summable, then $\Sigma f_p a_p$ is (C, r)-summable whenever Σa_p is (C, r)-summable.

To simplify Kojima's conditions, we note first that, for i fixed,

$$\lim_n \frac{A_{n-i+1}^{(s-k)}}{A_n^{(s)}} = \lim_n \frac{\binom{n-i+1+s-k-1}{s-k}}{\binom{n+s-1}{s}} = \begin{cases} 0 & \text{if } k > 0, \\ 1 & \text{if } k = 0. \end{cases} \tag{$*$}$$

Note that $(*)$ holds even if $s - k < 0$, since the numerator of the first

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fraction is 0 for this case. Using (*) we find that, for i fixed,

$$\lim_n\left[\frac{A_i^{(r)}}{A_n^{(s)}}\sum_{k=0}^{r+1}\binom{r+1}{k}A_{n-i+1}^{(s-k)}\Delta^{r-(k-1)}f_{i+k}\right]$$

$$=\sum_{k=0}^{r+1}\left[\binom{r+1}{k}A_i^{(r)}\Delta^{r-(k-1)}f_{i+k}\lim_n\frac{A_{n-i+1}^{(s-k)}}{A_n^{(s)}}\right]$$

$$=A_i^{(r)}\Delta^{r+1}f_i$$

$$=\binom{i+r-1}{r}\Delta^{r+1}f_i.$$

Thus if (2) holds, then

$$\sum_{i=1}^{\infty}\binom{i+r-1}{r}|\Delta^{r+1}f_i|<\infty. \tag{2'}$$

Note the special case of (2′) when $r = 0$:

$$\sum_{i=1}^{\infty}|\Delta f_i|<\infty, \tag{2''}$$

which means that $f = \{f_p\}$ is of bounded variation, and consequently is not only bounded but convergent.

Next we show that for $k \leqslant r$ and $1 \leqslant i \leqslant n$,

$$\frac{A_i^{(r)}A_{n-i+1}^{(s-k)}}{A_n^{(s)}}\leqslant (s!)(r!)A_{i+k}^{(r-k)}. \tag{**}$$

We note that (**) holds if $s < k$, since $A_{n-i+1}^{(s-k)} = 0$ for this case. For $2 \leqslant i \leqslant n$ and $0 < k \leqslant \min\{s, r\}$, we have

$$\frac{A_i^{(r)}A_{n-i+1}^{(s-k)}}{A_n^{(s)}}=\frac{(i+r-1)!}{r!\,(i-1)!}\;\frac{(n-i+1+s-k-1)!}{(s-k)!\,(n-i+1-1)!}\Big/\frac{(n+s-1)!}{s!\,(n-1)!}$$

$$=\frac{[(i+k-1)\cdots(i)][(n-1)\cdots(n-i+1)]}{(n+s-1)(n+s-2)\cdots(n+s-i-k+1)}$$

$$\cdot\frac{s!\,(r-k)!}{r!\,(s-k)!}\cdot\frac{(i+r-1)!}{(r-k)!\,(i+k-1)!}$$

$$\leqslant\frac{(n+k-1)(n+k-2)\cdots(n-i+1)}{(n+s-1)(n+s-2)\cdots(n+s-i-k+1)}$$

$$\cdot(s!)(r!)\binom{i+r-1}{r-k}$$

$$\leqslant (s!)(r!)A_{i+k}^{(r-k)}.$$

We omit similar proofs for the two cases $k = 0, i \geqslant 1$ and $k > 0, i = 1$.

Now suppose (1) and (2′) hold and $s \leqslant r$. Then $f \in BV_{r+1}$, where BV_j is the set of all bounded complex sequences x such that

$$\sum_{p=1}^{\infty} \binom{p+j-2}{j-1} |\Delta^j x_p| < \infty.$$

It is known [**1**, pp. 350–351] that if $x \in \mathrm{BV}_j$, then $x \in \mathrm{BV}_t$ for $1 \leq t \leq j-1$. Thus $\sum_{p=1}^{\infty} \binom{p+t-2}{t-1} |\Delta^t f_p| < \infty$ for $1 \leq t \leq r+1$, so that, using (**), we have

$$\begin{aligned}
&\frac{1}{A_n^{(s)}} \sum_{i=1}^{n-r-1} A_i^{(r)} \left| \sum_{k=0}^{r+1} \binom{r+1}{k} A_{n-i+1}^{(s-k)} \Delta^{r-(k-1)} f_{i+k} \right| \\
&\leq \sum_{i=1}^{n-r-1} \sum_{k=0}^{r} \binom{r+1}{k} \frac{A_i^{(r)} A_{n-i+1}^{(s-k)}}{A_n^{(s)}} |\Delta^{r-(k-1)} f_{i+k}| \\
&\leq \sum_{i=1}^{n-r-1} \sum_{k=0}^{r} \binom{r+1}{k} (s!)(r!) A_{i+k}^{(r-k)} |\Delta^{r-(k-1)} f_{i+k}| \\
&\leq (r+1)!\,(r!)(s!) \sum_{i=1}^{n-r-1} \sum_{k=0}^{r} \binom{i+k+r-k-1}{r-k} |\Delta^{r-(k-1)} f_{i+k}| \\
&\leq (r+1)!\,(r!)(s!) \sum_{k=0}^{r} \sum_{i=1}^{\infty} \binom{i+r-1}{r-k} |\Delta^{r-k+1} f_{i+k}|,
\end{aligned}$$

and consequently, since the last expression is free of n, we see that (2) holds.

Thus we have shown that if (1) and (2′) hold and $s \leq r$, then (2) holds, and we previously showed that (2) implies (2′) in any case. Hence we have the following result.

THEOREM 1. *In order for f to have the property that $\sum f_p a_p$ is (C, s)-summable whenever $\sum a_p$ is (C, r)-summable, $s \leq r$, it is necessary and sufficient that* (1) *and* (2′) *hold.*

The following special case of Theorem 1 is especially useful for our purpose:

In order for f to have the property that $\sum f_p a_p$ is (C, r)-summable whenever $\sum a_p$ is (C, r)-summable, it is necessary and sufficient that $f \in \mathrm{BV}_{r+1}$.

THEOREM 2. *If $\sum f_p a_p$ is (C, s)-summable whenever $\sum a_p$ is (C, r)-summable, then $\sum f_p a_p$ is (C, r)-summable whenever $\sum a_p$ is (C, r)-summable.*

PROOF. The statement is obvious if $s \leq r$. Suppose $r < s$. From Theorem K and a previous argument, we see that (2′) holds. Now since the set of convergent sequences ($(C, 0)$-summable sequences) is a subset of the set of (C, r)-summable sequences, we know that $\sum f_p a_p$ is (C, s)-summable whenever $\sum a_p$ is $(C, 0)$-summable. Hence from Theorem K and a previous argument, (2″) holds, so that f is bounded. But f bounded and (2′) imply that $f \in \mathrm{BV}_{r+1}$, and this, by the special case of Theorem 1 mentioned above, means that $\sum f_p a_p$ is (C, r)-summable whenever $\sum a_p$ is (C, r)-summable.

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