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AN EXTREMAL PROBLEM FOR POLYNOMIALS WITH A PRESCRIBED ZERO. II

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ABSTRACT. Improving upon an earlier estimate it is shown that if $p_n(z)$ is a polynomial of degree at most n such that $p_n(1) = 0$ and $\max_{|z|=1}|p_n(z)| \leqslant 1$, then $|p_n(0)| < 1 - (1.03369)/n + O(1/n^2)$.

Let $\mathcal{P}_n$ be the set of all polynomials $p_n(z)$ of degree at most n satisfying $p_n(1) = 0$ and $\max_{|z|=1}|p_n(z)| \leqslant 1$. The problem of determining

$$\mu(n) := \sup_{p_n(z)\in\mathcal{P}_n} |p_n(0)| \tag{1}$$

was raised in [3, p. 364, Problem 8.2]. Its solution would have applications in the power sum theory of P. Turán.

It is known that there exist positive constants c_1, c_2 such that

$$1 - c_1/n \leqslant \mu(n) \leqslant 1 - c_2/n;$$

hence for large n there is a best possible estimate

$$\mu(n) \leqslant 1 - c/n + o(1/n), \tag{2}$$

where

$$c = \varliminf_{n\to\infty} n\cdot(1-\mu(n))$$

is an absolute constant.

The first upper estimate which appeared in print [5] was

$$c \leqslant \frac{1}{\pi}\int_{-\infty}^{\infty}\left|\log\left(1 - \frac{\sin^2 u}{u^2}\right)\right|du = 2.55132\ldots.$$

Subsequently we showed in [4] that

$$c \leqslant \pi^2/8 = 1.23370\ldots.$$

Various ways of arriving at a lower estimate for c are known, but none of them leads to anything better than $c \geqslant 1$. For example, if $p_n(z) \in \mathcal{P}_n$, then by an inequality of Callahan [1]

$$\left(\frac{1}{2\pi}\int_0^{2\pi}|p_n(e^{i\theta})|^2 d\theta\right)^{1/2} \leqslant \left(\frac{n}{n+1}\right)^{1/2},$$

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whereas by a result in [2, Corollary 3]

$$|p_n(0)| \leq \left(\frac{n}{n+1}\right)^{1/2}\left(\frac{1}{2\pi}\int_0^{2\pi}|p_n(e^{i\theta})|^2\,d\theta\right)^{1/2},$$

so that

$$\mu(n) \leq \frac{n}{n+1} = 1 - \frac{1}{n} + O\left(\frac{1}{n^2}\right).$$

We would arrive at the same estimate if we used the formula

$$p_n(0) = \frac{1}{n+1}\sum_{\nu=0}^{n} p_n(e^{i2\pi\nu/(n+1)}) \tag{3}$$

and took into account the fact that the term corresponding to $\nu = 0$ vanishes. For this reason several people were inclined to believe that c in (2) may in fact be equal to 1. The purpose of this paper is to show that c is certainly larger than 1. Indeed, we have the following

THEOREM. *For the constant c in* (2) *the lower estimate $c > 1.03369\ldots$ holds.*

PROOF. It is easily seen that there exists a polynomial $p_n^*(z) \in \mathcal{P}_n$ such that $\mu(n) = p_n^*(0) > 0$.

Now let $l_\nu(z)$ be the fundamental functions of Lagrange interpolation with respect to the points $e^{i\theta_\nu}$ where

$$\theta_\nu := \frac{2\pi\nu}{n+1} \qquad (\nu = 0, 1, 2, \ldots, n),$$

i.e.

$$l_0(z) = \frac{1}{n+1}\,\frac{z^{n+1}-1}{z-1}, \quad l_\nu(z) = l_0(e^{-i\theta_\nu}z) \qquad (\nu = 1, 2, \ldots, n). \tag{4}$$

Then $p_n^*(z)$ may be represented as

$$p_n^*(z) = \sum_{\nu=0}^{n} p_n^*(e^{i\theta_\nu})l_\nu(z).$$

Now let us write $p_n^*(z)$ in the form

$$p_n^*(z) = q_n(z) + r_n(z), \tag{5}$$

where

$$q_n(z) := \sum_{\nu=1}^{n} l_\nu(z) = 1 - \frac{1}{n+1}\,\frac{z^{n+1}-1}{z-1},$$

$$r_n(z) = \sum_{\nu=1}^{n} \delta_\nu l_\nu(z),$$

and

$$\delta_\nu = p_n^*(e^{i\theta_\nu}) - 1. \tag{6}$$

Next we observe that

$$t(\theta) := \sum_{\nu=0}^{n} |l_\nu(e^{i\theta})|^2 \equiv 1. \tag{7}$$

In fact, $t(\theta)$ is a trigonometric polynomial of degree at most n which, because of (4), has a period of length $2\pi/(n+1)$. Considering the Fourier series development of $t(\theta/(n+1))$ we readily see that $t(\theta)$ must be a constant, and hence $t(\theta) \equiv t(0) = 1$.

Using the Cauchy-Schwarz inequality in conjunction with (7) we obtain

$$\max_{0 \leqslant \theta \leqslant 2\pi} |r_n(e^{i\theta})| \leqslant \left(\sum_{\nu=1}^{n} |\delta_\nu|^2 \right)^{1/2}.$$

Thus, from (5) we get

$$1 \geqslant \max_{|z|=1} |p_n^*(z)| \geqslant \max_{|z|=1} |q_n(z)| - \left(\sum_{\nu=1}^{n} |\delta_\nu|^2 \right)^{1/2},$$

and consequently

$$\left(\sum_{\nu=1}^{n} |\delta_\nu|^2 \right)^{1/2} \geqslant \max_{|z|=1} |q_n(z)| - 1. \tag{8}$$

By a simple geometric consideration we deduce from (6) that

$$\operatorname{Re} p_n^*(e^{i\theta_\nu}) \leqslant 1 - |\delta_\nu|^2/2.$$

Hence (3) and (8) give us

$$\begin{aligned} \mu(n) = p_n^*(0) &= \frac{1}{n+1} \sum_{\nu=0}^{n} \operatorname{Re} p_n^*(e^{i\theta_\nu}) \leqslant \frac{1}{n+1} \sum_{\nu=1}^{n} \left(1 - \frac{|\delta_\nu|^2}{2} \right) \\ &\leqslant 1 - \frac{1}{n} - \frac{1}{2n} \left(\max_{|z|=1} |q_n(z)| - 1 \right)^2 + O\left(\frac{1}{n^2} \right). \end{aligned} \tag{9}$$

Finally, taking $\alpha = 4.085573885$ we obtain by a numerical calculation

$$\begin{aligned} |q_n(e^{i\alpha/n})| &= \left\{ \left(1 - \frac{\sin \alpha}{\alpha} \right)^2 + \left(\frac{1 - \cos \alpha}{\alpha} \right)^2 \right\}^{1/2} + O\left(\frac{1}{n} \right) \\ &= 1.259590522 \ldots + O\left(\frac{1}{n} \right). \end{aligned}$$

Using this value as a lower estimate for $\max_{|z|=1}|q_n(z)|$ in (9) we arrive at the desired result.

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