A SIMPLER PROOF THAT COMPACT METRIC SPACES ARE SUPERCOMPACT

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ABSTRACT. We give a simpler proof that every compact metric space is supercompact.

In 1969 De Groot [5] introduced the notion of supercompactness (see definition below). All supercompact spaces are compact by the Alexander subbase lemma; the converse of this was recently shown false by Bell [1].

This leaves the question of what hypotheses on a space, beyond compactness, enable one to conclude that it is supercompact. Strok and Szymański [7] have shown, in particular, that all compact metric spaces are supercompact; their proof, however, is very complicated.

The purpose of this note is to offer a simpler proof. We have been advised that our proof is similar to one of van Douwen [4], though ours is somewhat simpler.

The author has recently shown [6] that all compact groups are supercompact.

A collection $\mathcal{L}$ is said to be linked if whenever $A, B \in \mathcal{L}$, $A \cap B \neq \emptyset$. $\mathcal{B}$ is binary if for every linked $\mathcal{L} \subseteq \mathcal{B}$, $\bigcap \mathcal{L} \neq \emptyset$. A space $X$ is said to be supercompact if $X$ has a binary closed subbase.

Fix a compact metric space $X$ and let $\{C_n : n \in \omega\}$ be a closed base for $X$. We shall construct a sequence $\{S_n : n \in \omega\}$ of finite families of closed sets such that for each $n \in \omega$,

1. $\bigcup_{m < n} S_m = C_n$,
2. $\bigcup_{m < n} S_m$ is binary.

By (1), $S = \bigcup_{n \in \omega} S_n$ is a subbase for $X$; since $X$ is compact, (2) implies that $S$ is binary.

We proceed by induction on $n$. Set

$$S = \bigcup_{m < n} S_m.$$ 

Observe that by induction hypothesis $S$ is finite and binary. For $p \in C_n$, let

$$\# p = |\{F \in S : p \in F\}|$$

and let

$$A_k = \{x \in C_n : \# x \geq k\}.$$
Note that there is $k_0$ such that $A_{k_0} = \emptyset$.

We define by downward induction on $k$ a finite set $\mathcal{G}_k$ of closed subsets of $X$ such that

1. $A_k \subset \bigcup \mathcal{G}_k \subset C_n$;
2. $\mathcal{G}_k \cup \mathcal{F}$ is binary;
3. $\bigcup \mathcal{G}_k$ is a neighborhood (relative to $C_n$) of $A_k$;
4. For every $\mathcal{A} \subset \mathcal{F}$, either $\bigcap \mathcal{A} \setminus \bigcup \mathcal{G}_k$ is infinite or $\bigcap \mathcal{A} \cap (C_n \setminus \bigcup \mathcal{G}_k)$ is finite.

By (1) and (2), we may take $\mathcal{F}_n = \mathcal{G}_0$, so it remains only to construct the $\mathcal{G}_k$'s. Take $\mathcal{G}_{k_0} = \emptyset$. Fix $k < k_0$ and assume $\mathcal{G}_{k+1}$ has been constructed. Set $C = C_n \setminus \bigcup \mathcal{G}_{k+1}$;

let $\mathcal{K}$ be the set of all intersections with $C$ of intersections of precisely $k$ distinct members of $\mathcal{F}$. By (3), $\mathcal{K}$ is a disjoint collection; also, $\mathcal{K}$ is finite. For each $\mathcal{A} \subset \mathcal{F}$ such that $\bigcap \mathcal{A}$ has a limit point in $C \setminus \bigcup \mathcal{K}$, pick $x_\mathcal{A}$ to be such a limit. Since $\mathcal{F}$ is finite and $X$ is normal, there is a neighborhood $N$ of $\bigcap \mathcal{K}$ such that $N \cap \{x_\mathcal{A} : \mathcal{A} \subset \mathcal{F} \& \bigcap \mathcal{A} \text{ has a limit in } C \setminus \bigcup \mathcal{K}\} = \emptyset$.

If $\mathcal{G}_k$ is chosen so that $\bigcup \mathcal{K} \subset \bigcup \mathcal{G}_k \subset N$, (4) will be satisfied. Since $\mathcal{F} \cup \mathcal{G}_{k+1}$ is finite and $X$ is normal, we may pick a finite disjoint cover $\mathcal{Q}$ of $\bigcup \mathcal{K}$ consisting of:

1. For each infinite $H \in \mathcal{K}$, a neighborhood $U_H$ of $H$ such that $\overline{U_H}$ meets no members of $\mathcal{F} \cup \mathcal{G}_{k+1}$ that do not meet $H$.
2. For each finite $H \in \mathcal{K}$ and each $p \in H$, a neighborhood $U_p$ of $p$ such that $U_p$ meets no members of $\mathcal{F} \cup \mathcal{G}_{k+1}$ that do not contain $p$.

Set $\mathcal{G}_k = \{U \cap N \cap C : U \in \mathcal{Q}\}$ if $G = \overline{U_H} \cap N \cap C$ (respectively $G = \overline{U_p} \cap N \cap C$). Set $G = G_H$ (respectively $G = G_p$).

If $\mathcal{L} \subset \mathcal{G}_k \cup \mathcal{G}_{k+1}$ and $\mathcal{L}$ is linked, then since $\mathcal{G}_k$ is disjoint, $\mathcal{L}$ contains at most one member of $\mathcal{G}_k$ of $\mathcal{G}_0$. If $G = G_H$ for some $H \in \mathcal{K}$, pick $p_\mathcal{L} \in H \setminus \bigcup \mathcal{G}_{k+1}$ in such a way that $p_\mathcal{L} = p_\mathcal{L}$ only if $\mathcal{L} = \mathcal{L}$; this is possible since (by (4)) $H \setminus \bigcup \mathcal{G}_{k+1}$ is infinite. For $G \in \mathcal{G}_{k+1}$ let $G' = G \cup \{p_\mathcal{L} : G \in \mathcal{L}\}$.

Observe that $G' \setminus G$ is finite, so $G'$ is closed. Set $\mathcal{G}_k = \mathcal{G}_k \cup \{G' : G \in \mathcal{G}_{k+1}\}$. It is clear that (1), (3), and (4) hold of $\mathcal{G}_k$; it remains only to prove that $\mathcal{G}_k \cup \mathcal{F}$ is binary. Let $\mathcal{E} \subset \mathcal{G}_k \cup \mathcal{F}$ be linked.

**Claim.** $\mathcal{L}' = \{G \in \mathcal{G}_{k+1} : G' \in \mathcal{L}\}$ is linked. For assume that $G_0, G_1 \in \mathcal{L}'$. Since $G_0 \cap G_1 \neq \emptyset$, either $G_0 \cap G_1 \neq \emptyset$ or there is $p_\mathcal{L} \in G_0 \cap G_1$. But this implies that $G_0, G_1 \in \mathcal{L}''$; since $\mathcal{L}''$ is linked, $G_0 \cap G_1 \neq \emptyset$.

There are three cases to consider.

**Case 1.** $\mathcal{L} = \{F_0, \ldots, F_n\} \cup \{G_0, \ldots, G_m\}$ for some $F_0, \ldots, F_n \in \mathcal{F}$.
Then $\cap \mathcal{L} \supseteq F_0 \cap \cdots \cap F_n \cap \mathcal{L}'$ which is nonempty by induction hypothesis.

**Case 2.** $\mathcal{L} = \{G_0\} \cup \{F_0, \ldots, F_n\} \cup \{G_0, \ldots, G_m\}$.
Then by construction (of $G_p$), $p \in \cap \mathcal{L}$.

**Case 3.** $\mathcal{L} = \{G_H\} \cup \{F_0, \ldots, F_n\} \cup \{G_0, \ldots, G_m\}$.
By choice of $G_H$, each $F_i$ contains $H$; in particular, $p_{e_F} \in F_0 \cap \cdots \cap F_n$. Then $p_{e_F} \in \cap F_i$.  \[\square\]

REFERENCES

1. M. G. Bell, Not all compact Hausdorff spaces are supercompact, General Topology and Appl. 8 (1978), 151–155.
6. C. F. Mills, Compact groups are supercompact (to appear).

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