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FLAT PARTIAL CONNECTIONS AND HOLOMORPHIC STRUCTURES IN C^∞ VECTOR BUNDLES

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ABSTRACT. The notion of a flat partial connection D in a C^∞ vector bundle E, defined on an integrable subbundle F of the complexified tangent bundle of a manifold X is defined. It is shown that E can be trivialized by local sections s satisfying $Ds = 0$. The sheaf of germs of sections s of E satisfying $Ds = 0$ has a natural fine resolution, giving the de Rham and Dolbeault resolutions as special cases.

If X is a complex manifold and F the tangents of type (0, 1), the flat partial connections in a C^∞ vector bundle E are put in correspondence with the holomorphic structures in E.

If X, E are homogeneous and F invariant, then invariant flat connections in E can be characterized as extensions of the representation of the isotropic subgroup to which E is associated, extending results of Tirao and Wolf in the holomorphic case.

1. Introduction. Let E be a holomorphic vector bundle over a complex manifold X and $TX^{\mathbf{C}} = F \oplus \bar{F}$ the splitting of the tangent bundle of X into subbundles of types (0, 1) and (1, 0) respectively. Then F is closed under Lie bracket, and there is a unique first order differential operator

$$D\colon \Gamma E \to \Gamma F^* \otimes E \tag{1}$$

such that

$$D(fs) = fDs + \bar{\partial} f \otimes s \tag{2}$$

for f in $C^\infty(X)$, s in ΓE and where $Ds = 0$ on an open set U if and only if s is holomorphic on U. If we put

$$\nabla_\xi s = (Ds)(\xi), \qquad \xi \in \Gamma F, \tag{3}$$

identifying $F^* \otimes E$ with $\operatorname{Hom}(F, E)$, then ∇_ξ behaves like a covariant derivative in E, but is only defined for ξ in ΓF. Moreover ∇ is flat:

$$\nabla_{[\xi,\eta]} = \nabla_\xi \nabla_\eta - \nabla_\eta \nabla_\xi, \qquad \xi, \eta \in \Gamma F. \tag{4}$$

If we begin with a C^∞ vector bundle E over X and a differential operator D as in (1), satisfying (2), we can ask if E always has a holomorphic structure such that the solutions of $Ds = 0$ are precisely the (local) holomorphic sections of E. The answer is yes, provided the operators ∇ defined by (3)

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satisfy (4). This is the Corollary to the Theorem (see below). It is useful to encode the holomorphic structure as such an operator D or ∇, since operations on the category of C^∞ vector bundles often extend automatically to the category of C^∞ vector bundles with connections (or partial connections).

For example, if K and Q are line bundles with i: $Q^2 \to K$ an isomorphism and ∇^K a partial connection in K, there is a unique partial connection ∇^Q in Q such that

$$\nabla^K_\xi i(s_1 \otimes s_2) = i\big(\nabla^Q_\xi s_1 \otimes s_2 + s_1 \otimes \nabla^Q_\xi s_2\big)$$

for all ξ for which ∇^K is defined and all sections s_1, s_2 of Q. Then ∇^Q is flat if and only if ∇^K is. If K is holomorphic and ∇^K the partial connection on the (0, 1) tangent bundle defined above, ∇^K is flat and hence Q has a flat partial connection on the (0, 1) tangents. Then Q has a holomorphic structure and the isomorphism i: $Q^2 \to K$ becomes an isomorphism of holomorphic line bundles; cf. [**5**].

Partial connections were introduced by Bott [**2**] for foliations. We can combine the real foliation, and complex structure versions as follows: If X is any manifold, $TX^{\mathbf{C}}$ the complexified tangent bundle, a subbundle $F \subset TX^{\mathbf{C}}$ is integrable if

(i) $F \cap \bar{F}$ has constant rank,

(ii) F and $F + \bar{F}$ are closed under Lie bracket.

Then according to Nirenberg [**7**], X can be covered by open sets U on which there are coordinates $u_1, \ldots, u_k, v_1, \ldots, v_l, x_1, \ldots, x_m, y_1, \ldots, y_m$ where, if $z_j = x_j + \sqrt{-1}\ y_j, j = 1, \ldots, m$, F is spanned on U by

$$\partial/\partial u_1, \ldots, \partial/\partial u_k, \partial/\partial \bar{z}_1, \ldots, \partial/\partial \bar{z}_m.$$

If f is a C^∞ function on X, let d^Ff denote the restriction of df to F, regarded as a section of the dual bundle F^*. Putting $\Omega^p_F = \Gamma\Lambda^pF^*$, d^F extends to a differential

$$d^F: \Omega^p_F \to \Omega^{p+1}_F, \qquad p \geqslant 0,$$

with all the usual properties, including a local Poincaré Lemma (see [**8**]).

Let E be a C^∞ vector bundle over X, then a partial connection defined on F, or an F-connection is a linear map

$$D: \Gamma E \to \Gamma F^* \otimes E$$

satisfying

$$D(fs) = fDs + d^Ff \otimes s$$

for all f in $C^\infty(X)$, s in ΓE. D extends to a map

$$D: \Omega^p_F(E) \to \Omega^{p+1}_F(E), \qquad p \geqslant 0,$$

where $\Omega^p_F(E) = \Gamma\Lambda^pF^* \otimes E$. $D \circ D$ defines a section R of $\Lambda^2F^* \otimes \mathrm{End}(E)$ which is the curvature, and we say D is flat if $R = 0$.

An example of a flat F-connection may be obtained by generalizing Bott's construction. Let $F^0 \subset T^*X^{\mathbf{C}}$ be all covectors vanishing on F. We define

$$D\colon \Gamma F^0 \to \Gamma F^* \otimes F^0$$

by

$$(Ds)(\xi) = \xi \lrcorner ds, \qquad s \in \Gamma F^0, \xi \in \Gamma F. \tag{5}$$

This makes sense since s is a 1-form on X. Since $\xi \lrcorner s = 0$ for all ξ in ΓF, s in ΓF^0, the right-hand side of (5) is the Lie derivative of s with respect to ξ. This shows D is flat. In the case $F = \bar{F}$, F is the tangent bundle to a foliation, F^0 the (co-) normal bundle and D is Bott's connection along the leaves of F.

THEOREM 1. *A C^∞ vector bundle E admits a flat F-connection D, where F is integrable, if and only if it can be trivialized (locally) by sections s satisfying $Ds = 0$.*

COROLLARY. *Let X be a complex manifold, F the bundle of tangents of type* (0, 1) *and E a C^∞ vector bundle with a flat F-connection D then E has a unique holomorphic structure such that the holomorphic sections on an open set U are the solutions of $Ds = 0$ on U.*

The corollary is an immediate consequence of Theorem 1. Theorem 1 is proven in §2, and further applications in §3. Operators such as D are examples of overdetermined systems considered in [**4**]. In the case at hand a simple direct proof of Theorem 1 can be given using estimates from [**6**], its only being necessary to check that these estimates imply smooth dependence on parameters.

A version of these results for line bundles already appears in [**8**] with applications to Kostant's theory of geometric quantization.

N. J. Hitchin, in joint work with M. F. Atiyah and I. M. Singer, has an alternative proof of the corollary [**1**], and I would like to thank him for several useful conversations on this topic. I would also like to thank J. T. Lewis for his valuable help and advice in the preparation of this paper.

2. Proof of Theorem 1. $F \cap \bar{F}$ is real and integrable. We can choose, through any given point, x, a submanifold Y of X transversal to the leaves of $F \cap \bar{F}$. Then $F' = F|Y$ satisfies $F' \cap \bar{F}' = 0$ and is integrable. If we solve the problem on Y we can parallelly translate the sections along the leaves of $F \cap \bar{F}$ and so obtain a solution in a neighbourhood of x. Thus we may assume $F \cap \bar{F} = 0$.

If $F \cap \bar{F} = 0$ we have coordinates $v_1, \ldots, v_l, x_1, \ldots, x_m, y_1, \ldots, y_m$, with F spanned by $\partial/\partial\bar{z}_1, \ldots, \partial/\partial\bar{z}_m$ where $z_j = x_j + \sqrt{-1}\, y_j$, $j = 1, \ldots, m$, as before. Choose a local frame field $s_1, \ldots, s_N$ for E on this coordinate neighbourhood and define matrices A_j of functions by

$$\nabla_{\partial/\partial\bar{z}_j} s_b = \sum_{a=1}^{N} (A_j)_{ab} s_a, \qquad b = 1, \ldots, N, j = 1, \ldots, m.$$

Then ∇ is flat if

$$\partial A_j/\partial\bar{z}_i - \partial A_i/\partial\bar{z}_j + [A_i, A_j] = 0, \qquad i, j = 1, \ldots, m. \tag{6}$$

Put $A = \sum_{j=1}^m A_j d\bar{z}_j$ and regard $v_1, \ldots, v_l$ as parameters, then equations (6) become

$$\bar{\partial}A + A \wedge A = 0. \tag{7}$$

This is the formal integrability condition for having a matrix g of functions which is invertible and satisfying

$$\bar{\partial}g + Ag = 0. \tag{8}$$

It is shown in [6] that, when there are no parameters, (8) always has a solution provided (7) holds. We shall check that the proof of [6] goes through with smooth dependence on parameters so that g is a C^∞ function of $v_1, \ldots, v_l$, $x_1, \ldots, x_m, y_1, \ldots, y_m$. Then we may define

$$t_b = \sum_{a=1}^{N} g_{ab}s_a, \qquad b = 1, \ldots, N,$$

and obtain a frame field $t_1, \ldots, t_N$ satisfying

$$Dt_a = 0, \qquad a = 1, \ldots, N.$$

This will complete the proof of the theorem.

The proof in [6] uses an explicit homotopy operator T for $\bar{\partial}$ in a polycylinder of radius R in the coordinates $z_1, \ldots, z_m$:

$$\beta = T\bar{\partial}\beta + \bar{\partial}T\beta$$

for every (p, q)-form β with $q > 0$. A Hölder norm $\|\cdot\|$ is defined on forms on this polycylinder and it is shown that

$$\|T\| \leqslant c_1 R$$

for some constant $c_1 > 0$. Moreover, $\|A\|$ depends continuously on $v_1, \ldots, v_l$ (as parameters) and, restricting them to a fixed compact neighbourhood, we have

$$\|A\| \leqslant c_2$$

uniformly in $v_1, \ldots, v_l$.

Thus the operator $f \mapsto T(Af)$ on matrices of functions satisfies

$$\|T(Af)\| \leqslant c_1 c_2 R \|f\|,$$

and by choosing R so that $c_1 c_2 R < 1$, a contraction mapping is obtained. Then if g is a solution of

$$g = \psi - T(Ag) \tag{9}$$

where $\bar{\partial}\psi = 0$, we have

$$\begin{aligned}\bar{\partial} g &= -\bar{\partial} T(Ag) = -Ag + T\big(\bar{\partial}(Ag)\big)\\ &= -Ag + T\big((\bar{\partial} A)g\big) - T\big(A \wedge \bar{\partial} g\big)\\ &= -Ag - T\big(A \wedge (\bar{\partial} g + Ag)\big).\end{aligned}$$

Thus

$$\bar{\partial} g + Ag = T\big(A \wedge (\bar{\partial} g + Ag)\big)$$

and, since $T \circ A$ is a contraction, we must have

$$\bar{\partial} g + Ag = 0.$$

Moreover, by the uniqueness of fixed points, g depends on $v_1, \ldots, v_l$. By choosing ψ suitably, and modifying T as in [6] we can make sure g is invertible at x, and hence in a neighbourhood of x (when we have established continuity).

g depends differentiably on $x_1, \ldots, x_m, y_1, \ldots, y_m$ if A does, from the proof in [6], when $v_1, \ldots, v_l$ are held fixed. To see that it also depends differentiably on $v_1, \ldots, v_l$, we observe that formally differentiating (9) gives

$$\partial g/\partial v_i = T(\partial A/\partial v_i g) - T(A \partial g/\partial v_i), \tag{10}$$

an equation of the same kind as (9).

Solutions to both equations (9) and (10) are obtained iteratively by setting

$$g_{n+1} = \psi + T(Ag_n)$$

and $g = \lim_{n\to\infty} g_n$. If formal differentiation inside T is allowed, the sequence $\partial g_n/\partial v_i$ converges for each i, uniformly in $v_1, \ldots, v_l$ and so the limit exists and is continuous by standard results in analysis. T is built from integral operators acting on the variables $z_1, \ldots, z_m$ one by one. It suffices, therefore, to consider the case $m = 1$. We abbreviate $v_1, \ldots, v_l$ by v. The operators are of the form

$$(Kf)(v, z) = \oint_{|\zeta| = R} f(v, \zeta)/(\zeta - z)\, d\zeta, \quad (Lf)(v, z)$$

$$(Lf)(v, z) = \iint_{|\zeta| < R} f(v, \zeta)/(\zeta - z)\, d\zeta \wedge d\bar{\zeta}.$$

The first clearly causes no difficulties when $|z| < R$. In the second, the methods of [3, p. 21] allow the integrand to be improved to

$$(L'f)(v, z) = \iint_{|\zeta| < R} f(v, \zeta) \log |\zeta - z|^2\, d\zeta \wedge d\bar{\zeta}.$$

But $\log |\zeta - z|$ is the fundamental solution of the Laplacian in the plane, and standard results from potential theory, see for example [9], imply that $L'f$ is C^∞ if f is. Thus T maps C^∞ forms to C^∞ forms. This concludes the proof.

3. Applications. Let E be a C^∞ vector bundle over X, $F \subset TX^{\mathbf{C}}$ an integrable subbundle and D a flat F-connection. Define $\Omega_F^p(E)$ as in the introduction. Let $\mathcal{Q}^p$ denote the sheaf associated to the presheaf $U \mapsto \Omega_F^p(E|U)$ and D the induced map from $\mathcal{Q}^p$ to $\mathcal{Q}^{p+1}$. Let $\mathcal{S}$ be the sheaf of germs of solutions of $Ds = 0$ which is a subsheaf of $\mathcal{Q}^0$, then we have a sequence

$$0 \to \mathcal{S} \hookrightarrow \mathcal{Q}^0 \xrightarrow{D} \mathcal{Q}^1 \xrightarrow{D} \mathcal{Q}^2 \to \dots . \tag{11}$$

THEOREM 2. (11) *is a fine resolution of* $\mathcal{S}$ *so that the cohomology groups* $H^p(\mathcal{S})$ *are isomorphic to those of the complex*

$$\Omega_F^0(E) \xrightarrow{D} \Omega_F^1(E) \xrightarrow{D} \Omega_F^2(E) \to \dots .$$

PROOF. The sheaves $\mathcal{Q}^p$ are clearly fine, and $D \circ D = 0$ since D is flat. It remains to prove that if $D\beta = 0$, β in $\Gamma(\mathcal{Q}^p, U)$, $p > 0$, for some open set U then for each x in U there is an open set V in U containing x and α in $\Gamma(\mathcal{Q}^{p-1}, V)$ with $\beta|V = D\alpha$. But by Theorem 1 there is an open set W in U containing x which has a local frame field $s_1, \dots, s_N$ for E with $Ds_i = 0$. Then, on W,

$$\beta = \sum_{a=1}^{N} \beta_a \otimes s_a$$

with $\beta_a \in \Omega^p_{F|W}$, and $D\beta = 0$ implies $d^F\beta_a = 0$. But $\beta_a = d^F\alpha_a$ on some common neighbourhood V of x, by the Poincaré lemma for d^F and then $\alpha = \sum_{a=1}^N \alpha_a \otimes s_a$ gives the required section.

REMARK. Theorem 2 contains the usual de Rham and Dolbeault isomorphisms as special cases by taking F real or $TX^{\mathbf{C}} = F \oplus \bar{F}$.

A second application generalizes some of the results of **[10]**. Let $X = G/H$ be a homogeneous space for a Lie group G, and let $F \subset TX^{\mathbf{C}}$ be invariant. Then there is a subspace $\mathfrak{p} \subset \mathfrak{g}^{\mathbf{C}}$ ($\mathfrak{g}$ the Lie algebra of G) containing $\mathfrak{h}$ and $\mathrm{Ad}_G H$-stable, which, when translated around X from the identity coset, gives F. F is closed under Lie bracket if and only if $\mathfrak{p}$ is a subalgebra, and F is integrable if, in addition, $\mathfrak{p} + \bar{\mathfrak{p}}$ is a subalgebra. Let E be a homogeneous vector bundle over X and $g \cdot s$, g in G, s in ΓE the induced action of G on sections of E. Let $g \cdot \xi$ denote the induced action on sections of F, then an F-connection D in E is invariant if

$$g \cdot (\nabla_\xi s) = \nabla_{g \cdot \xi} g \cdot s$$

for ξ in ΓF, s in ΓE and g in G.

If we differentiate the action of G on ΓE we get a representation of $\mathfrak{g}$ and we extend it to $\mathfrak{g}^{\mathbf{C}}$ by linearity. For $a \in \mathfrak{g}^{\mathbf{C}}$ let ξ^a be the vector field it determines on X so that F_{eH} is generated by $\{\xi^a_{eH} | a \in \mathfrak{p}\}$. Then for $a \in \mathfrak{p}$ we have two operations on ΓE, namely $a \cdot s$ and $\nabla_{\xi^a} s$. Moreover

$$a \cdot (fs) = fa \cdot s - \xi^a(f)s$$

for f in $C^\infty(X)$, s in ΓE and a in $\mathfrak{g}^{\mathbf{C}}$. Thus

$$a \cdot s + \nabla_{\xi^a} s, \qquad a \in \mathfrak{p},$$

is $C^\infty(X)$-linear in s and hence, by evaluation at the identity coset eH, defines an endomorphism $\gamma(a)$ of E_{eH}. For a in $\mathfrak{h}$ this is the derivative of the action of H on E_{eH} defining E, since $\xi^a = 0$.

PROPOSITION. *γ determines D uniquely, and D is flat if and only if γ is a representation of $\mathfrak{p}$ on E_{eH}.*

The details of the proof are straightforward and are left to the reader. **[10]** dealt with the case where X was complex and E holomorphic. In the Kostant-Kirillov-Dixmier programme for constructing representations, subalgebras $\mathfrak{p}$ (polarizations) arise which do not correspond with complex structures. Further applications of the notion of flat partial connections will appear elsewhere.

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