A CHARACTERIZATION OF PRIME NOETHERIAN P.I. RINGS AND A THEOREM OF MORI-NAGATA

AMIRAM BRAUN

ABSTRACT. Let $R$ be a noetherian prime p.i. ring, $C$ the center of $R$ and $\overline{C}$ its normalization. It is proved that $R$ is integral over its center iff $\overline{C}$ is a Krull domain. We also give a simple proof for the following theorem [7]: The normalization of a commutative noetherian domain is Krull.

Introduction. The purpose of this note is twofold. Firstly, to give an elementary simple proof of the so-called "Mori-Nagata" theorem [7]: "Let $C$ be a noetherian commutative domain, then $\overline{C}$--its normalization, is Krull." A generalization for the nonnoetherian case is also obtained. Secondly, we give a characterization for a noetherian prime P.I. (polynomial identity) ring $R$ to be integral over its center. One should note that there already exists a fairly simple proof for the Mori-Nagata theorem [8]. Unfortunately, this proof does not seem to be useful in the noncommutative case.

Our main theorem in §2 is that "A noetherian prime P.I. ring $R$ is integral over its center $C$, iff $\overline{C}$ is Krull." This result may be viewed as a generalization of the following theorem ([1], [2]): "A noetherian prime P.I. ring $R$ is finite over its center $C$, iff $C$ is noetherian." The special case of $C = \overline{C}$ is already obtained in [11].

The commutative results are handled in §1, while the noncommutative part is in §2. The characterization obtained in §2 is a straightforward application of the methods in §1, hence we shall merely mention the difference between these proofs. All rings are with unit, and for notations we follow [5] in §1 and [9] in §2.

1. All rings are commutative. We shall give here a short elementary proof and a mild generalization to the following theorem.

THEOREM [7]. Let $C$ be a noetherian integral domain, then $\overline{C}$ is a Krull domain.

For a proof of the theorem the following result due to Matijevic is important.

PROPOSITION 1.1 [6] $C$ as above, $K = qf(C)$. Let $C' = \{x \in K | \exists I an ideal in C with xI \subseteq C and C/I is artinian\}$. Then every subring $S, C \subseteq S \subseteq C'$ is noetherian.
 Remark 1.2. For a very short proof and a noncommutative generalization see [10].

Corollary 1.3 (Krull-Akizuki [7]). Let $C$ be a noetherian domain, $\dim(C) = 1$ then every overring $S, C \subset S \subset K = \text{qf}(C)$ is noetherian.

We construct a sequence of rings $\{C_{\lambda}\}, \lambda$ an ordinal number, by means of transfinite induction. They will satisfy the following properties:

(i) $C \subset C_{\lambda} \subset \bar{C}$,
(ii) $\forall x \in C_{\lambda} \exists m_1 = m_1(x), \ldots, m_r = m_r(x)$ with $xm_1^{e_1} \cdots m_r^{e_r} \subset C$, where $m_i, i = 1, \ldots, r$, are maximal ideals of $C$ and $e = e_1(x), \ldots, e_r = e_r(x)$ are in $\mathbb{N}$.

Say $C_1 = C$. And assume $C_{\lambda}$ is defined. If for all maximal ideals $p$ in $C_{\lambda}$ either $\text{grad}(p) \neq 1$, or $\text{grad}(p) = 1$ and $p \cdot p^{-1} = C_{\lambda}$ happens, we stop the process in the $\lambda$ step. Otherwise, we take $p$ maximal ideal in $C_{\lambda}$ with $\text{grad}(p) = 1$ and $p \cdot p^{-1} = p$ we define $C_{\lambda+1} = p^{-1}$. It is easy to check that $C_{\lambda+1}$ is a ring. If $\mu$ is a limit ordinal and $C_{\lambda}$ is defined $\forall \lambda < \mu$, we define $C_{\mu} \equiv \bigcup_{\lambda < \mu} C_{\lambda}$.

Remark 1.4. (a) For $\rho < \nu$ we have $C_{\rho} \subset C_{\nu}$.
(b) Each $C_{\rho}$ is noetherian.
Proof. (a) is easy.
(b) combines Proposition 1.1 and property (ii).

Lemma 1.5. With the previous notation $C_{\mu}$ satisfies (i) and (ii).

Proof. Say $\mu = \lambda + 1$, then $p \cdot p^{-1} = p$ and $C_{\lambda}$ is noetherian implies that $p^{-1} \subset \bar{C}$. For (ii) let $x \in C_{\lambda+1}$, $xp \subset p \subset C_{\lambda}$ and so $x(p \cap C) \subset C_{\lambda}$. Let $p \cap C = (a_1, \ldots, a_s)$. Then $xa_i \in C_{\lambda}$ for $i = 1, \ldots, s$. Pick $v_1, \ldots, v_s$ maximal ideal in $C$ with $(xa_i)v_1^i \cdots v_s^i \subset C$ for $i = 1, \ldots, s$ and then $x(p \cap C)v_1^i \cdots v_s^i \subset C$ but $p \cap C$ is a maximal ideal in $C$ since $p$ is in $C_{\lambda}$.

Properties (i) and (ii) are easily checked for $\mu$ a limit ordinal. Q.E.D.

Theorem 1.6. Let $C$ be a noetherian domain then $\bar{C}$ is Krull.

Proof. $C$ is noetherian implies that $C = \cap C_q$ where the intersection runs on all maximal ideal ones $q$ and the intersection is locally finite [5]. Hence $\bar{C} = C^* = \cap C^* = \cap C_q$ [4] ($C^*-$the complete integral closure of $C$), and obviously the intersection is still locally finite. Hence it suffices to show that each $\bar{C}_q$ is Krull; that is, we may assume that $K$-$\dim C < \infty$ to begin with. We shall prove the theorem via induction on Krull-$\dim(C)$. The case $K$-$\dim C = 1$ is established in Corollary 1.3. We apply now the process of constructing $\{C_{\lambda}\}$ from $C$. If the process does not stop we exhaust $\bar{C}$ after a step $\mu$ with $\text{card}(\mu) > \text{card}(\bar{C})$, and so $\bar{C} = C_{\mu}$ is noetherian, hence Krull.

Say the process stops at $\mu$; then $\forall p$ a maximal ideal in $C_{\mu}$ either $\text{grad}(p) \neq 1$ or if $\text{grad}(p) = 1, p \cdot p^{-1} = C_{\mu}$ and then $h(p) = 1$. Now, $C_{\mu}$ is noetherian;
hence $C_p = \cap_q (C_p)_q$ where grad($q$) = 1 and the intersection is locally finite. But

$$\dim(C_p)_q < \dim C_p = \dim C$$

for each such $q$ hence by induction $(C_p)_q$ is Krull. Again,

$$\bar{C} = \bar{C}_p = C_p^* = \cap (C_p)_q^* = \cap (C_p)_q$$

and the intersection is locally finite, hence $\bar{C}$ is Krull. Q.E.D.

The next lemma, definitely well known, is left as an exercise to the reader.

**Lemma 1.7.** Let $C$ be a domain, $S \subseteq C$ a multiplicative closed set, and $p_S$ is a prime ideal in $C_S$ with grad($p_S$) = 1, then grad($p$) = 1.

**Theorem 1.8.** Let $C$ be a domain such that $C_p$ is noetherian for all prime ideals $p$ in $C$, then

1. $C = \cap \bar{C}_p$, the intersection is on all prime ideals $\bar{p}$ in $\bar{C}$ with $h(\bar{p}) = 1$;
2. $C_p$ is a D.V.R. for all $\bar{p}$, $h(\bar{p}) = 1$ where $\bar{p}$ is a prime ideal in $\bar{C}$.

**Proof.** We observe first that $h(\bar{p}) < \infty$ for all prime ideals $\bar{p}$ in $\bar{C}$. Indeed let $p = \bar{p} \cap C$ then $K$-dim $C_p < \infty$, and so $h(\bar{p}) < K$-dim $\bar{C}_p < \infty$. We prove (1) by induction on $h(\bar{p})$. More precisely, we prove inductively that for $\bar{p}$ prime ideal in $\bar{C}$ and grad($\bar{p}$) = 1, $\bar{C}_\bar{p} = \cap \bar{C}_p$, where $\bar{p}$ are prime ideals in $\bar{C}$ and $h(\bar{p}) = 1$. Given $\bar{p}$ in $\bar{C}$, grad($\bar{p}$) = 1. Let $S = \bar{C} \setminus \bar{p}$, $S_1 = C \setminus p$. If grad($p_S$) = 1 by Lemma 1.7, grad($p_S$) = 1, but

$$\bar{p}_S, \cap (\bar{C})_{S_1} = (\bar{C}_{S_1}) = \bar{C}_p$$

is Krull (Theorem 1.6) hence $h(\bar{p}_{S_1}) = 1$ and consequently $h(\bar{p}) = 1$. We may therefore assume that grad($\bar{p}_S$) > 1 and so $\bar{C}_\bar{p} = \cap (\bar{C}_{S_1}_W, W \setminus \bar{C}_S$, and grad($W$) = 1. Clearly $W = V_S$ where $V$ is an ideal in $\bar{C}$ and $V \subseteq \bar{p}$. Now grad($W$) = 1 = grad($V_S$) and we apply again Lemma 1.7 to get that grad($V$) = 1. We can apply our induction hypothesis on $V (h(V) < h(\bar{p}))$ and so $\bar{C}_V = \cap \bar{C}_{\bar{q}}$, $h(\bar{q}) = 1$, $\bar{q}$ are prime ideals in $\bar{C}$. All in all $(\bar{C}_p)_W = \bar{C}_V = \cap \bar{C}_{\bar{q}}$, $h(\bar{q}) = 1$, and so $\bar{C}_\bar{p} = \bar{C}_S = \cap \bar{C}_q$, $h(\bar{q}) = 1$, $\bar{q}$ is a prime ideal of $\bar{C}$.

To prove (2) we let $\bar{p} \subseteq \bar{C}$ be a prime ideal in $\bar{C}$ and, $h(\bar{p}) = 1$ and $p = \bar{p} \cap C$, $S_1 = C \setminus p$. Then $\bar{p}_{S_1}$ is a prime ideal of $\bar{C}_{S_1}$ and $h((\bar{p})_{S_1}) = 1$, and $\bar{C}_{S_1}$ being Krull implies that $(\bar{C}_{S_1})_{\bar{p}_{S_1}} = \bar{C}_p$ is a D.V.R. Q.E.D.

The next theorem is a direct consequence of the preceding one.

**Theorem 1.9.** Let $C$ be a domain satisfying the following conditions:

1. $C_p$ is noetherian for every prime ideal $p$ in $C$.
2. $C = \cap_q C_q$, where $q$ is a maximal grade one prime ideal and the intersection is locally finite.

Then $C$ is Krull.
Proof. That $C = \cap_q C_q$ and the intersection is locally finite implies that $C^* = \cap C_q^*$ [4], moreover, $C_q$ is noetherian; hence, $C_q^* = \overline{C_q} = \cap \overline{C_p}$ where $h(\overline{p}) = 1$, $\overline{p}$ is a prime ideal in $\overline{C}$ and $\overline{p} \cap C \subseteq q$ for some $q$. We claim that every prime $\overline{p}$ in $\overline{C}$, with $h(\overline{p}) = 1$ appears in this intersection. Indeed, let $\overline{p}$ be such ideal and $p \equiv \overline{p} \cap C$ then $\text{grad}(\overline{p} \cap C_p) = 1$ [7]; hence $\text{grad}(p_p) = 1$ and so $\text{grad}(p) = 1$ (1.7), but then $p \subset q$, $q$ a maximal ideal of grade one. Thus $\overline{p_q}$ is a prime ideal in $\overline{C_q}$ and $h(\overline{p_q}) = 1$ and so $(\overline{C_q})_{\overline{p_q}} = \overline{C_p}$ appears in the intersection $\bigcap (\overline{C_q})_{\overline{W}}, h(\overline{W}) = 1$, $\overline{W}$ is prime in $\overline{C_q}$. Condition (1) and 1.8 guarantees that $\overline{C} = \cap \overline{C_p}, h(\overline{p}) = 1$ and $\overline{p}$ is prime in $\overline{C}$, hence $\overline{C} = C^*$; that is, $\overline{C}$ is Krull.

2. We switch now to the noncommutative analog of §1 in order to prove the following:

**Theorem 2.1.** Let $R$ be a noetherian prime P.I. ring, then $R$ is integral over its center $C$ iff $\overline{C}$ is a Krull domain.

The proof will follow closely the pattern of the previous section and so we shall merely emphasize the places where an additional argument is required. We begin by stating three lemmas, the proofs of which are given later.

**Lemma 2.2.** Let $R$ be as in Theorem 2.1, then $C = \cap p C_p$ where the intersection runs on all maximal grade one prime ideals $p$, and the intersection is locally finite.

**Lemma 2.3** [11]. $R$ as in Theorem 2.1. Let $x \in K$—the quotient field of $C$ and $xM \subseteq M$ where $M = m_1 R + \cdots + m_k R$ a right $R$ module and $m_i r = rm_i$ for each $r$ in $R$, then $x$ is integral over $C$.

**Lemma 2.4.** $R$ as in Theorem 2.1, $p$ a prime ideal in $C$ then $h(p) < \infty$.

**Corollary 2.5.** Let $R$ be as in Theorem 2.1, then $\overline{C} = C^*$—the complete integral closure of $C$.

Proof. Let $x \in C^*$ and $a \in C$ with $ax^n \in C$ for $n = 1, 2, \ldots$. Then $x^n \in R/a$ for $n = 1, 2, \ldots$, and so $R \subset R[x] \subset R1/a$. $R$ being noetherian implies that $R[x] = Rv_1 + \cdots + Rv_r$ and we may take $v_i$ to be a polynomial in $x$, hence $v_i \in Z(R[x])$. We apply Lemma 2.3 with $M = R[x]$ and get that $x \in \overline{C}$, the other inclusion is trivial.

**Proof of Theorem 2.1.** We define via transfinite induction the sequence $\{R_\lambda\}$, obeying the following conditions:

(1) $R \subseteq R_\lambda \subseteq R\overline{C}$.

(2) For $x$ in $R_\lambda$, $\exists m_1, \ldots, m_\nu$ maximal ideals in $C$, and $i_1, \ldots, i_\nu$, such that $x m_1^{i_1} \cdots m_\nu^{i_\nu} \subseteq R$.

(3) $R_\lambda$ is integral over $C_\lambda$ and $C_\lambda \subseteq \overline{C}$ where $C_\lambda = Z(R_\lambda)$. We observe first that once $R_\lambda$ is defined then $R_\lambda$ is left and right noetherian [10]. Given an ordinal $\mu$ and assume that $R_\lambda$ is defined for each $\lambda < \mu$ we define $R_\mu$. 


Case (i). \( \mu = \lambda + 1 \) then \( R_\mu \equiv m_0^{-1} R_\lambda \) where \( m_0 \) is a maximal ideal in \( C_\lambda \), \( \text{grad}(m_0) = 1 \) and \( m_0^{-1} m_0 = m_0 \). The absence of such maximal \( m_0 \) will stop the process and we would not define \( R_\mu \).

Case (ii). \( \mu \) is a limit ordinal then \( R_\mu = \lambda < \mu \) \( R_\lambda \). It is easy to check that \( R_\mu \) is a ring and that (1) is satisfied. To check (2) we see first that for \( \mu \) a limit ordinal it is trivial. If \( \mu = \lambda + 1 \) we pick \( a_1, \ldots, a_r \) in \( C \) such that \( mR = a_1R + \cdots + a_rR \) where \( m = m_0 \cap C \) and then \( x_{a_i} \in R_\lambda \) for \( i = 1, \ldots, r \), and there are \( m_1, \ldots, m_r \) such that \((x_{a_i})_{m_{1\cdots m_r}} m_j^y \subseteq R \) and so \( x_{m_{1\cdots m_r}} m_j^y \subseteq R \), \( m \) is maximal in \( C \) since \( C_\lambda \subseteq \bar{C} \).

To check (3) we see again that \( \mu \) is a limit ordinal is easy. Indeed, let \( x \in R_\mu \) then \( x \in R_\lambda \) for \( \lambda < \mu \), \( x \) is integral over \( C_\lambda \subseteq C_\mu \). Say \( \mu = \lambda + 1 \) then \( R_{\lambda+1} = m_0^{-1} R_\lambda \subseteq R_\lambda / y \) where \( y \in m_0 \) for some \( y \), then \( R_\lambda \) being noetherian implies

\[
R_{\lambda+1} = R_\lambda w_1 + \cdots + R_\lambda w_l
\]

and we can take \( w_i \in m_0^{-1} \subseteq Z(R_{\lambda+1}) \). Let \( x \in C_{\lambda+1} \) then by Lemma 2.3 \( x \) is integral over \( C_\lambda \) and in particular \( w_i, i = 1, \ldots, l \), are so. Let \( y \in R_{\lambda+1} \) then \( y = \Sigma t_i w_i, i = 1, \ldots, l \). But \( t_i, \ldots, t_l \in R \) are integral over \( C_\lambda \) and we have that

\[
D = C_\lambda[t_1, \ldots, t_l, w_1, \ldots, w_l]
\]

is a finite \( C_\lambda \) module [9], now \( y \in D \) and so \( y \) is integral over \( C_\lambda \) by the determinant argument. This also establishes the inclusion \( C_\lambda \subseteq C_{\lambda+1} \subseteq \bar{C} \).

The proof follows now the same lines as in §1 via induction on \( \dim R = \dim C < \infty \), the reduction to this case is by Lemmas 2.2 and 2.4, the case \( \dim R = 1 \) is valid by Remark 1.2. We need two additional remarks. First, if the \( R_\lambda \) sequence does not stop, then \( R_\lambda = R\bar{C} \) for some \( \mu \) and \( R_{\mu+1} \) is noetherian and integral over its center \( C_\mu \subseteq \bar{C} \); but then \( \bar{C} = C_\mu \) and \( RC \) is centrally integrally closed in the sense of [11] and hence \( \bar{C} \) is Krull [11].

Secondly, the following remark is needed: Let \( q < C_\lambda \) with \( \text{grad}(q) = 1 \), \( q \) is a maximal ideal and \( q^{-1} q = C_\lambda \) then \( h(q) = 1 \). Indeed, we may localize by \( q \), if \( p \subseteq q \), prime ideal in \( C_\lambda \), then \( (q^{-1} p)q \subseteq p \), hence \( q^{-1} p \subseteq p \), hence \( q^{-1} p = p \) or \( p = qp \), hence \( pR_\lambda = q(pR_\lambda) \). But \( q \in J(R_\lambda) \) since \( q \) is maximal and \( R_\lambda \) is integral over \( C_\lambda \) and so we get a contradiction via Nakayama’s Lemma. Q.E.D.

We have to prove the converse: Let \( R \) be a noetherian prime P.I. ring and \( \bar{C} \) is Krull, then \( R \) is integral over \( C \). From [11] we have that \( C^* \supset Z(T(R)) \) where \( T(R) \) is the trace-envelope of \( R \) and so \( (\bar{C})^* \supset Z(T(R)) \) but \( (\bar{C})^* = \bar{C} \) (\( \bar{C} \) being Krull) implies that \( \bar{C} \supset Z(T(R)) \). Thus for \( r \in R, c_r(r), i = 1, \ldots, n \), the coefficients of its characteristic polynomial are integral over \( C \) and so \( r \) is integral over \( C \). Q.E.D.

Note 2.6. One can get a better criterion by merely requiring that \( \bar{C} = \cap C_i, h(\bar{C}) = 1 \) and \( C_i \) is a D.V.R. The proof, though, is more complicated.
Moreover, if \( \overline{C} \) is completely integrally closed, no noetherian condition is needed to show that \( R \) is integral over \( C \) [11].

Remark 2.7. There is an example [11] of a noetherian prime P.I. ring, integral over its center and is not a finite module over it.

We need to prove Lemmas 2.2, 2.3, and 2.4.

Proof of Lemma 2.2. That \( C = \cap_p C_p \) where \( \text{grad}(p) = 1 \) is always true [5]. We prove that the intersection is locally finite. We follow Kaplansky’s commutative proof [5]. Firstly, if \( p \) is maximal with respect to the principal ideal \( (x) \) then there exists a prime ideal \( P \) in \( R \) and \( y \in C \) such that \( P \supseteq \{ r \in R | yr \in xR \} \) and \( y \not\in xC \) and so \( p = \{ z \in C | zy \in xC \} \) where \( p = P \cap C \) [11]. We show that every regular sequence in \( p \) is of length one. \( \{ x \} \) is obviously a maximal sequence. Let \( z \in p \) be another element, we shall show that \( p \subseteq Z(C/zC) \). We have \( py \subseteq xC \), \( y \not\in xC \) implies \( z \cdot y \in xC \) or \( zy = xz \). We claim that \( v \not\in zC \) and \( pv \subseteq zC \). Indeed if \( v = zw \), \( zy = xzw \) or \( y = zw \) a contradiction. Also, \( xpv = (xz)v = zyp \subseteq zC \); hence, \( pv \subseteq zC \). To finish the proof, let \( x \in p \), \( \text{grad}(p) = 1 \) and \( p \) is maximal over \( y \). Then by the previous argument we have that \( p \subseteq Z(C/xC) \). If \( \exists p_1 \not\supseteq p \) prime and maximal via \( p_1 \subset Z(C/xC) \) then \( p_1 \subset Z(C/yC) \) contradicting the maximality of \( p \), hence \( p \) is maximal of grade one over \( x \). There are only finitely many such ideals [11], hence \( x \) belongs to only finitely many. Q.E.D.

Proof of Lemma 2.3. This is an immediate corollary of [11].

Proof of Lemma 2.4. First we prove the following:

Lemma (a). Let \( R \) be a noetherian prime P.I. ring, integral over a central subring \( D \). Then \( D \) satisfies the basic Principal Ideal Theorem.

Proof. Let \( p \) be a prime ideal in \( D \) minimal above \( x \in p \) and let \( P \) be a prime ideal in \( R \) with \( P \cap R = p \) (there are such since \( R \) is integral over \( D \)), then \( P \) is minimal over \( x \) (\( R \) is integral over \( D \)) and so \( \text{dim}(p) = 1 \) [3]. This implies that \( \text{dim } C_p = \text{dim } R_p = 1 \). Q.E.D.

We proceed now to prove the generalized Principal Ideal Theorem [5] for \( D \). The proof is almost identical with the one in [5].

Lemma (b). Let \( R \) be as in Lemma (a), then \( C = Z(R) \) satisfies the generalized Principal Ideal Theorem.

Proof. Let \( p \) be a prime ideal in \( C \) minimal above \( (a_1, \ldots, a_n) \) we prove via induction on \( n \) that \( h(p) < n \). The case \( n = 1 \) is provided by Lemma (a). We localize \( C \) at \( p \) and assume that \( C \) is local with \( p \) the maximal ideal. Say \( h(p) > n \) and let \( p = p_0 \supseteq p_1 \supseteq p_2 \supseteq \cdots \supseteq p_{n+1} \) a proper chain of length \( n + 1 \) and no primes between \( p \) and \( p_1 \) (we can do it since \( R \) is integral over \( C \) and noetherian). Obviously \( (a_1, \ldots, a_n) \not\subseteq p_1 \) say \( a_i \not\in p_1 \) then \( p \) is the only minimal prime above \( (a_1, p_1) \) and therefore \( p_1/(a_1, p_1) \) is a nil ideal in \( C/(a, p_1) \), and we can find \( i \) such that

\[
a_i^t = c_ia_i + b_i, \quad i = 2, \ldots, n,
\]
and \( b_i \in \pi, c_i \in C \). Let \( J = (b_2, \ldots, b_n) \subseteq \pi \). We find by induction \( q \subseteq \pi \) with \( q \supset J \) and \( h(q) < n - 1 \). Let \( Q \) be a prime ideal of \( R \) with \( Q \cap C = q \). Then \( R/Q \) is integral over \( C/q \) but \( p/q \) a minimal prime over \((a_i + q)/q\) is not of height one, a contradiction to Lemma (a).

To complete the proof of Lemma 2.4 one should observe that for \( p \) a prime ideal in \( C, pR = a_1R + \cdots + a_tR \) and that \( p \) is minimal above \((a_1, \ldots, a_t)\).

REFERENCES


DEPARTMENT OF MATHEMATICS, RUTGERS UNIVERSITY, NEW BRUNSWICK, NEW JERSEY 08903