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ASYMPTOTIC STABILITY OF $\mathrm{Ass}(M/I^nM)$

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ABSTRACT. Let R be a noetherian ring, I an ideal of R and M a finitely generated R-module. Let A be the map from $\mathbf{N}$ into the subsets of $\mathrm{Spec}(R)$ defined by $A(n) = \mathrm{Ass}_R(M/I^nM)$. We shall prove:

(1) For n sufficiently large, $A(n)$ is independent of n.

We moreover give two examples concerning the behavior of $A(n)$.

Apparently the first attempt to study the map A (in the case $M = R$) was made by D. Rees [2], who proved that P belongs to infinitely many sets $A(n)$, if it is prime divisor of the integral closure of a power of I. Ratliff [1] improved this, in showing that the above condition even implies that P belongs to almost all sets $A(n)$. He moreover showed that for fixed I, $M = R$, there is a $k \in \mathbf{N}$, such that $P \in A(km)$ implies $p \in A(kn)$ for all large n, and conjectured that, in case R is a domain, k may be chosen to be 1 [1, PD_0], e.g.

(2) *Conjecture*. If $R = M$ is a domain, $P \in A(k)$ for some k implies $P \in A(n)$ for all large n.

We shall see that this fails even for affine k-domains.

Now we prove (1). The argument occurring in [1, proof (2.11)] also applies in the case of arbitrary M to show that $L = \bigcup A(n)$ is finite. But the finiteness of L implies that (1) is shown, once we have proven: There is a $p \in \mathbf{N}$, such that $A(m) - A(m-1) \subseteq A(n)$ for all m and n with $p < m \leqslant n$. In the notations

$$M_n = I^nM/I^{n+1}M, \quad B(n) = \mathrm{Ass}_R(M_n) \qquad (n \geqslant 0)$$

the short exact sequences $0 \to M_n \to M/I^{n+1}M \to M/I^nM \to 0$ show that

$$A(n+1) - A(n) \subseteq B(n) \subseteq A(n+1) \qquad (n \geqslant 1). \tag{3}$$

So (1) is established, if we may prove that $B(n)$ is increasing for large n. To show this, let us first assume that I contains an M-regular element, and accept the following result, which also occurs in [1] for $M = R$.

(4) LEMMA. *If* $((0): I)_M = (0)$, *there is an* $h \in \mathbf{N}$ *such that*

$$(I^{n+1}M: I)_M = I^nM \quad \textit{for all } n \geqslant h.$$

Now let $n \geqslant h - 1$ and $P \in B(n)$, and let us show that $P \in B(n+1)$. As localization at P does not affect the statement of (4), we may assume that

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(R, P) is local. $P \in B(n)$ then implies that there is an $x \in I^nM - I^{n+1}M$ such that $Px \subseteq I^{n+1}M$, hence $PIx \subseteq I^{n+2}M$. On the other hand (4) induces that $x \not\subseteq I^{n+2}M$, and we conclude that $P \in B(n+1)$.

To prove the general case we apply the above argument to I^tM instead of M, which is allowed by the following reduction result.

(5) There is a $t \in \mathbf{N}$ such that $((0)\colon I)_I t_M = (0)$.

To prove (5), put $N = ((0)\colon I)_M$. Then, by Artin-Rees, there is a t with $((0)\colon I)_{I^tM} \subseteq IN = (0)$.

(6) Remark. The above reduction argument was pointed out to the author by D. Eisenbud and simplified the original proof, which used a cohomological argument to make the reduction to the case $((0)\colon I)_M = (0)$.

Next we give a proof of (4) which is more direct than the corresponding one in [**1**].

As $\mathrm{Ass}_R(M/(I^{n+1}M\colon I)_M) \subseteq A(n+1) \subseteq L$, L being finite, we may restrict ourselves to prove the result for all localizations at $P \in L$, hence assume that R is local. Introduce the following associated graded objects:

$$\tilde{M} = \bigoplus_{n \geq 0} M_n, \qquad \tilde{R} = \bigoplus_{n \geq 0} R_n,$$

and let F_n be the set of leading forms in $\tilde{M}$ of all elements in $I^{n+1}M\colon I$ outside of I^nM. We must show that F_n vanishes for all large n. All the sets F_n are contained in the homomogeneous submodule

$$T = \bigoplus_{n \geq 0} T_n = ((0)\colon R_1)_{\tilde{M}}$$

of $\tilde{M}$. Let $t_1, \ldots, t_r$ be generating n forms of T, all of degree $< d$. Then for $n \geqslant d$, $T_n \subseteq R_1(t_1, \ldots, t_r)\tilde{R} = (0)$. This shows that T is quasi-(0) (e.g. $T_n = (0)$ for large n), and so it suffices to show that the minimal degree d_n of all elements of F_n converges with n to infinity. So choose an M-regular element $a \in I$. Then, by Artin-Rees, there is an $s \in \mathbf{N}$, such that for all $n > s$ we have

$$(I^{n+1}M\colon I)_M \subseteq (I^{n+1}M\colon a)_M = I^{n+1}(M/a) \cap M \subseteq I^{n-1}M,$$

hence $d_n \geqslant n - s$.

The following statement may help to decide whether a given prime P belongs to the asymptotic value A^* of A.

(7) If I contains an M-regular element, then A and B coincide for large arguments.

Indeed, by (3) we only have to show that $P \in A(n+1)$ implies $P \in B(n)$ for large values of n. In localizing, we may assume that (R, P) is local. Then $P \in A(n+1)$ and (4) imply for large n, $I^{n+1}M \subsetneqq I^{n+1}M\colon P \subseteq I^nM$, hence that $P \in B(n)$.

Using $B(n) \subseteq A(n+1)$ and the fact that $B(n)$ is increasing for large n we also see that B has an asymptotic value B^*.

We next give an example with $A^* \neq B^*$.

(8) Example. Put $R = M = K[X, Y, Z]/(XY, XZ, X^2)$, $I = YZR$, $P = (X, Y, Z)R$, where K is a field and X, Y, Z are indeterminates. Let J_n be the

preimage (XY, XZ, X^2, Y^nZ^n) of I^n in $K[X, Y, Z]$. Then $X \notin J_n$ and $X(X, Y, Z) \subseteq J_n$ show that $P \in A(n)$ for all $n > 0$. Assume that $P \in B(n)$ for an $n > 0$. Then there is an $f \in K[X, Y, Z]$ such that $Y^nZ^nf \notin J_{n+1}$, but $(X, Y, Z)Y^nZ^nf \subseteq J_{n+1}$. As $(X, YZ)Y^nZ^n \subseteq J_{n+1}$, we may assume that X as well as YZ do not occur in f. Thus we may write $f = u + v$, with $u \in K[y]$ and $v \in ZK[z]$. But now $Y^{n+1}Z^nf$, $Y^{n+1}Z^nv \in J_{n+1}$ imply $Y^{n+1}Z^nu \in J_{n+1} \cap K[Y, Z] = (Y^{n+1}Z^{n+1})$, hence that $u \in K[Y] \cap ZK[Y, Z] = (0)$. In the same way we get that $v = 0$. Thus, there is no f as above, which shows that $P \notin B(n)$ for all $n > 0$.

Now we show that (2) is not true.

(9) EXAMPLE. Let K be as above and put $R' = K[X, Y]$, $P = X^{m+1}R' + yR'$, $R = K + P$. Then P is a maximal ideal of R, and $K[X^{m+1}, Y] \subseteq R \subseteq R'$ shows that R is of finite type over K. R'/P obviously is a $K = R/P$-space of dimension $m + 1$, which shows that X does not satisfy an integral equation of degree lower than $m + 1$ over R. As $PR' = P$ we find $c, d \in P$ with $X = d/c$. Put $I = cR + dR$, $M = R$. We claim

$$P \in A(n) \Leftrightarrow n < m.$$

Obviously we have $I^n = \sum_{i \leqslant n} c^{n-i}d^iR = c^n\sum_{i \leqslant n} X^iR$, thus $I^n = c^nR'$ for $n \geqslant m$. This induces $A(n) \subseteq \{Q \cap R \mid Q \in \mathrm{Ass}_{R'}(R'/c^nR')\}$. The prime divisors of c^nR' are all nonmaximal, and so may not retract to P. This shows the "$\Rightarrow$"-part of the above statement.

Assume now that $P \notin A(n)$ for $n < m$. As none of the prime divisors of c^nR' retracts to P we find an $s \in P$ which is regular with respect to R/I^n and R'/c^nR'. By our choice of c and d we obviously get $c^nR' = I^nR'$, and $s \in P$ shows that $R'_s = R_s$. Thus we get $I^n = I^nR_s \cap R = I^nR'_s \cap R = c^nR' \cap R$. This induces that $d^nX = c^nX^{n+1} \in R \cap c^nR' = I^n$. We thus find $r_0, \ldots, r_n \in R$ with

$$d^nX = \sum_{i=0}^{n} d^ic^{n-i}r_i.$$

In dividing both sides of this equation by c^n, we get

$$X^{n+1} = \sum_{i=0}^{n} X^ir_i,$$

contrary to the fact that X may not satisfy an integral equation of degree lower than $m + 1$ over R.

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