ON \(^*-\)PRIMITIVE RINGS

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Abstract. Ortiz has defined a new radical for rings, called the \(\mathcal{K}\)-radical, which in general lies strictly between the prime radical and the Jacobson radical. In this paper a simple internal characterization of \(\mathcal{K}\)-primitive rings is given, and it is shown that among the \(\mathcal{K}\)-primitive rings are prime Noetherian rings and prime rings which satisfy a polynomial identity. In addition an analogue of the density theorem is proved for \(\mathcal{K}\)-primitive rings.

Throughout, \(R\) will denote an associative ring, not necessarily with unity element. If \(N\) is a submodule of a right \(R\)-module \(M\), then \((N : M) = \{ a \in R | Ma \subset N \}\). As in [1], let \(K_R\) denote the class of all right \(R\)-modules \(M\) such that

1. \((0 : M)\) is a prime ideal of \(R\);
2. if \(TV\) is a submodule of \(M\) for which \((TV : M) = (0 : M)\), then \(TV = 0\).

Ortiz has shown that the property \(K_R = \emptyset\) is a radical property and that the \(\mathcal{K}\)-radical is \(\bigcap \{(0 : M) | M \in K_R\}\). A right \(\mathcal{K}\)-primitive ideal of \(R\) is an ideal \(P\) such that \(P = (0 : M)\) for some \(M \in K_R\), and a right \(\mathcal{K}\)-primitive ring is a ring in which \(0\) is a right \(\mathcal{K}\)-primitive ideal. Thus \(R\) is right \(\mathcal{K}\)-primitive if and only if \(R\) is a prime ring and \(K_R\) contains a faithful right \(R\)-module. Left \(\mathcal{K}\)-primitive, etc., are defined analogously, and in this paper terms such as \("\mathcal{K}\)-primitive"\) will always mean \("right \mathcal{K}\)-primitive"\).

**Proposition 1.** A prime ring \(R\) is \(\mathcal{K}\)-primitive if and only if \(R\) contains a right ideal \(I\) which is maximal with respect to the property \((I : R) = 0\).

**Proof.** Suppose \(R\) is \(\mathcal{K}\)-primitive and let \(M\) be a faithful right \(R\)-module in \(K_R\). Choose \(m \neq 0\) in \(M\); let \(I = \{ x \in R | mx = 0 \}\); and suppose \(a \in (I : R)\). If \(N = \{ n \in M | nRa = 0 \}\), then since \(m \in N\), \(N\) is a nonzero submodule of \(M\), whence there exists \(b \neq 0\) in \((N : M)\). Thus \(Mb \subset N\) and so \(MbRa = 0\), which yields \(bRa = 0\) and hence \(a = 0\) since \(R\) is prime. Therefore \(R\) is a right ideal of \(R\) satisfying \((I : R) = 0\). Now let \(J\) be any right ideal of \(R\) properly containing \(I\). Since \(mJ\) is a nonzero submodule of \(M\), there is an \(x \neq 0\) in \((mJ : M)\). Let \(r \in R\). Since \(mRx \subset Mx \subset mJ\), there exists \(y \in J\)
such that \( mrx = my \). Thus \( rx - y \in I \) and so \( rx = (rx - y) + y \in J \). Hence \( Rx \subseteq J \), and so \( I \) is maximal with respect to \( (I : R) = 0 \).

Conversely, assume \( I \) is a right ideal of \( R \) which is maximal with respect to \( (I : R) = 0 \), and let \( M \) be the right \( R \)-module \( R/I \). Since \( (0 : M) = (I : R) = 0 \), \( M \) is faithful. If \( N \) is a nonzero submodule of \( M \), then its inverse image \( J \) in \( R \) is a right ideal properly containing \( I \), and by the maximality of \( I \), \( (J : R) \neq 0 \). Since \( (N : M) = (J : R) \), we have \( M \subseteq K_R \), and so \( R \) is \( K \)-primitive.

**Corollary 1.** Every primitive ring is \( K \)-primitive.

**Corollary 2.** Every right Noetherian prime ring is \( K \)-primitive.

**Theorem 1.** If \( R \) is a right order in a simple Artinian ring \( Q \) with center \( F \) such that \( Q = RF \), then \( R \) is \( K \)-primitive.

**Proof.** \( Q \cong D_n \) for some division ring \( D \). Let \( V \) be an \( n \)-dimensional right vector space over \( D \); choose \( v \neq 0 \) in \( V \); and let \( I = \{ x \in R | vx = 0 \} \). If \( a \in (I : R) \), then

\[
Va = vQa = vRFa = vRaF \subseteq vIF = 0,
\]

whence \( a = 0 \). Thus \( I \) is a right ideal of \( R \) satisfying \( (I : R) = 0 \). Let \( J \) be a right ideal of \( R \) properly containing \( I \). Since \( vJ \) is a nonzero \( R \)-submodule of \( V \), \( vJF \) is a nonzero \( Q \)-submodule of \( V \) and hence \( V = vJF \). Choose \( x \in JF \) such that \( v = vx \). There exist \( a \) and \( b \) in \( R \) with \( b \) regular such that \( 1 - x = ab^{-1} \). Hence \( vab^{-1} = v(1 - x) = 0 \), and so \( va = 0 \), which means that \( a \in I \). Thus \( b = a + xb \in I + JF \subset JF \). Since \( JF \) is a right ideal of \( Q \) and contains an invertible element \( b \) of \( Q \), we have \( JF = Q \). We now write \( 1 = \sum c_\lambda \lambda \) where the \( c_\lambda \) are in \( J \) and the \( \lambda \) are in \( F \). We may also write \( \lambda = d_\lambda e^{-1} \) where the \( d_\lambda \) are in \( R \) and \( e \) is a regular element of \( R \). For any \( r \in R \) we have

\[
r = 1 \cdot r = \sum c_\lambda \lambda r = \sum c_\lambda r \lambda = \sum c_\lambda rd_\lambda e^{-1},
\]

whence \( re = \sum c_\lambda rd_\lambda \in J \). Thus \( Re \subseteq J \) and so \( (J : R) \neq 0 \). Therefore \( I \) is maximal with respect to \( (I : R) = 0 \), so by Proposition 1, \( R \) is \( K \)-primitive. \( \square \)

Using Theorem 1 and Posner's Theorem [2], we establish

**Corollary 3.** Every prime ring satisfying a polynomial identity over its centroid is both left and right \( K \)-primitive.

A special case of Theorem 1 is worth noting. If \( Q \) is a division ring—that is, if \( R \) is a right Ore domain—then the proof of Theorem 1 shows that the right ideal \( I \) is 0. We shall call a \( K \)-primitive ring strongly \( K \)-primitive in case the right ideal \( I \) which is maximal with respect to \( (I : R) = 0 \) is \( I = 0 \). Thus we have

**Corollary 4.** If \( R \) is a right Ore domain with right quotient ring \( D \) having center \( F \) such that \( D = RF \), then \( R \) is strongly \( K \)-primitive.
Several obvious questions arise concerning $K$-primitive rings:

1. Are left and right $K$-primitivity equivalent?

2. Converse of Corollary 4. Is every strongly $K$-primitive ring a right Ore domain which, together with the center of its right quotient ring $D$, generates $D$?

3. Is every right Ore domain $K$-primitive? Strongly $K$-primitive?

4. Can the hypothesis $Q = RF$ in Theorem 1 be removed? Equivalently, is every prime right Goldie ring $K$-primitive?

These questions are open except for the second part of 3: Any simple right Noetherian domain, not a division ring, is a right Ore domain but is not strongly $K$-primitive; it is $K$-primitive however, by Corollary 2. Also, question 2 can be answered partially by

**Proposition 2.** Every strongly $K$-primitive ring is a right Ore domain.

**Proof.** Let $a$ and $b$ be nonzero elements of the strongly $K$-primitive ring $R$. By maximality of the zero right ideal, every nonzero right ideal $I$ of $R$ satisfies $(I : R) \neq 0$. In particular $(bR : R) \neq 0$, so there exists $b_1 \neq 0$ in $R$ such that $Rb_1 \subset bR$. If $ab = 0$, then $aRb_1 \subset abR = 0$, a contradiction since $R$ is prime; thus $R$ has no zero divisors. Moreover, since $ab_1 \in Rb_1 \subset bR$, there exists $a_1 \neq 0$ in $R$ such that $ab_1 = ba_1$, so $R$ is a right Ore domain. □

In [1] Ortiz showed that every $K$-primitive ring can be embedded in a full ring of linear transformations of a vector space over a division ring. We shall investigate this embedding and in fact prove an analogue of Jacobson's density theorem for $K$-primitive rings. One formulation of density is the following: If $V$ is a vector space over a division ring $D$, then a subring $R$ of $\text{Hom}_D(V, V)$ is dense if and only if $V$ is irreducible and for every finite-dimensional subspace $W$ and every vector $u \in W$, $(u(0 : W) : V) = R$. A slight variation of this definition leads to the desired characterization of $K$-primitive rings. We first define $V$ to be $K$-irreducible if and only if $vRa = 0$ with $v \in V$ and $a \in R$ implies $v = 0$ or $a = 0$. A subring $R$ of $\text{Hom}_D(V, V)$ will be called $K$-dense if and only if $V$ is $K$-irreducible and for every finite-dimensional subspace $W$ and every vector $u \in W$, $(u(0 : W) : V) \neq 0$.

**Theorem 2.** If $R$ is a $K$-primitive ring, $V$ is a faithful module in $K_R$, and $\bar{V}$ is the quasi-injective hull of $V$, then $D = \text{Hom}_R(\bar{V}, V)$ is a division ring, $V$ is a vector space over $D$, and $R$ is a $K$-dense subring of $\text{Hom}_D(V, V)$. Conversely, if a ring $R$ is a $K$-dense subring of $\text{Hom}_D(V, V)$ for some vector space $V$ over a division ring $D$, then $R$ is $K$-primitive, $V \in K_R$, and $D = \text{Hom}_R(\bar{V}, V)$.

**Proof.** Assume first that $R$ is a $K$-primitive ring with $V$ a faithful module in $K_R$. Ortiz [1] has shown that $D$ is a division ring, that $V$ is a vector space over $D$, and that the mapping $a \to a'$ defined by $va' = va$ for $v \in V$ and $a \in R$ is an embedding of $R$ in $\text{Hom}_D(V, V)$. We must show that $R$ is $K$-dense. Let $a \neq 0$ be in $R$ and let $N = \{v \in V | vRa = 0\}$. $N$ is a submodule
of $V$ and if $N \neq 0$, then $(N : V) \neq 0$, whence there exists $b \neq 0$ in $R$ such that $VbRa \subset NRa = 0$. This implies that $bRa = 0$, a contradiction since $R$ is prime. Thus $N = 0$, that is, $V$ is $K$-irreducible. If we show that for every finite-dimensional subspace $W$ and every vector $u \notin W$, we have $u(0 : W) \neq 0$, then $u(0 : W)$, being a nonzero submodule of $V$, would satisfy $(u(0 : W) : V) \neq 0$, thereby proving that $R$ is $K$-dense. Suppose then that $W$ is a finite-dimensional subspace of smallest dimension for which there is a vector $u \notin W$ such that $u(0 : W) = 0$. If $W = 0$, then $u(0 : W) = uR \neq 0$, so $\dim W > 0$. Let $W = W_0 + wD$ where

$$\dim W_0 = \dim W - 1$$

and $w \notin W_0$. The mapping $T: w(0 : W_0) \to u(0 : W_0)$ defined by $(wa)T = ua$ for $a \in (0 : W)$ is well-defined since $wa_1 = wa_2$ with $a_1$, $a_2$ in $(0 : W_0)$ implies

$$a_1 - a_2 \in (0 : W_0) \cap (0 : w) = (0 : W)$$

and hence $u(a_1 - a_2) = 0$. Since $\overline{V}$ is quasi-injective, $T$ can be extended to $\lambda \in \text{Hom}_R(V, \overline{V}) = D$. For any $a \in (0 : W_0)$ we have

$$ua = (wa)T = (wa)\lambda = (w\lambda)a$$

and so $(u - w\lambda)(0 : W_0) = 0$. By minimality of $W$ we must have $u - w\lambda \in W_0$ and hence $u \in W_0 + wD = W$, a contradiction. This proves that $R$ is $K$-dense.

Conversely, assume that $R$ is a $K$-dense subring of $\text{Hom}_D(V, V)$ for some vector space $V$ over a division ring $D$. Suppose $A$ and $B$ are left ideals of $R$ such that $AB = 0$. Choose $a \neq 0$ in $A$ and $v \neq 0$ in $V$. For any $b \in B$ we have $vRaRb \subset vAB = 0$. The $K$-irreducibility of $V$ implies that $vRa \neq 0$, so choosing $r \in R$ such that $vra \neq 0$, we have $vraRb = 0$ and by $K$-irreducibility again we have $b = 0$. Thus $R$ is a prime ring. Suppose $N$ is a nonzero $R$-submodule of $V$ and choose $u \neq 0$ in $N$. Taking $W = 0$ we have

$$0 \neq (u(0 : W) : V) = (uR : V) \subset (N : V)$$

and hence $V \in K_R$ and $R$ is $K$-primitive. To show that $D = \text{Hom}_R(\overline{V}, \overline{V})$, let $\lambda \in D$. The mapping $v \to v\lambda$ can be extended to $\lambda' \in \text{Hom}_R(\overline{V}, \overline{V})$. We shall show that $\lambda'$ is unique and that identifying $\lambda$ with $\lambda'$ yields the desired conclusion. Suppose $\lambda'$ and $\lambda''$ are both extensions of $v \to v\lambda$. Then $\lambda' - \lambda''$ is in $\text{Hom}_R(\overline{V}, \overline{V})$ and since the latter is a division ring, either $\lambda' - \lambda''$ is one-to-one or $\lambda' - \lambda''$ is 0. Since

$$\ker(\lambda' - \lambda'') \supset V \neq 0,$$

we have $\lambda' = \lambda''$. Thus the identification of $\lambda$ with $\lambda'$ is well defined and embeds $D$ in $\text{Hom}_R(\overline{V}, \overline{V})$. To complete the proof we must show that every element of $\text{Hom}_R(\overline{V}, \overline{V})$ is $\lambda'$ for some $\lambda \in D$. Suppose $f \neq 0$ is in $\text{Hom}_R(\overline{V}, \overline{V})$; since $f$ is one-to-one, $0 \neq Vf \subset V$. Thus there exist $u, w$ in $V$ such that $u = wf \neq 0$. Suppose $u$ and $w$ are linearly independent over $D$. 
Then \((u(0 : w) : V) \neq 0\) so there exists \(a \neq 0\) in \(R\) such that

\[Va \subset u(0 : w) = (wf)(0 : w) = w(0 : w)f = 0,\]

a contradiction. Hence \(u\) and \(w\) are linearly dependent, say \(u = w\lambda\) for some \(\lambda \in D\). Then \(w(\lambda' - f) = 0\) and hence \(\lambda' - f\), not being one-to-one, must be 0.

**References**


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