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INTERSECTIONS OF COMMUTANTS WITH CLOSURES OF DERIVATION RANGES

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ABSTRACT. The norm closure of the set $\mathcal{A}_w(\mathcal{X}) = \bigcup \{\mathrm{Ran}(\delta_A)^{-w} \cap \{A\}': A \in \mathcal{L}(\mathcal{X})\}$, where δ_A denotes the inner derivation induced by the operator A, $\mathrm{Ran}(\delta_A)^{-w}$ is the weak closure of the range of δ_A and $\{A\}'$ is the commutant of A, is disjoint from the open dense subset $\mathcal{B}(\mathcal{X}) = \{T \in \mathcal{L}(\mathcal{X})$: T has a nonzero normal eigenvalue$\}$ for every complex Banach space $\mathcal{X}$. For a Hilbert space $\mathcal{H}$, $\mathcal{L}(\mathcal{H}) = \mathcal{B}(\mathcal{H}) \cup \mathcal{A}_w(\mathcal{H})^-$, where the bar denotes norm closure.

1. Introduction. Let $\mathcal{X}$ be a (nonzero) complex Banach space and let $\mathcal{L}(\mathcal{X})$ denote the algebra of all (bounded linear) operators acting on $\mathcal{X}$. To each $\mathcal{A} \in \mathcal{L}(\mathcal{X})$ we associate the following objects: δ_A (the inner derivation of $\mathcal{L}(\mathcal{X})$ defined by $\delta_A(X) = AX - XA$), $\mathrm{Ran}(\delta_A)^-$ (the norm closure of the range of δ_A), $\mathrm{Ran}(\delta_A)^{-w}$ (the closure of the range of δ_A in the weak operator topology), $\{A\}'$ (the commutant of A), $\sigma(A)$ (the spectrum of A) and $\mathrm{sp}(A)$ (the spectral radius of A).

In **[15]** Hong W. Kim analyzed the set

$$\mathcal{A} = \bigcup \left\{\mathrm{Ran}(\delta_A)^- \cap \{A\}': A \in \mathcal{L}(\mathcal{X})\right\}$$

for the case when $\mathcal{X} = \mathcal{H}$ is a Hilbert space and raised the following problems:

[15, Question 1]. Is $\mathcal{L}(\mathcal{X}) \setminus \mathcal{A}$ (norm) dense in $\mathcal{L}(\mathcal{X})$?

[15, Question 3]. Is every thin operator in $\mathcal{A}$?

(where $B \in \mathcal{L}(\mathcal{X})$ is *thin*, in the sense of A. Brown and C. Pearcy **[4]**, if $B = \lambda - K$ for some $\lambda \in \mathbf{C}$ = complex plane, identified with the multiples of the identity operator, and some $K \in \mathcal{K}$ = the ideal of compact operators). The subset of all thin operators will be denoted by (T).

The answer to the first question is yes. Indeed, more is actually true: $\mathcal{A}_w = \bigcup \{\mathrm{Ran}(\delta_A)^{-w} \cap \{A\}': A \in \mathcal{L}(\mathcal{X})\}$ is *nowhere dense* in $\mathcal{L}(\mathcal{X})$ for every complex Banach space $\mathcal{X}$.

The answer to the third question is no: If $\lambda - K \in \mathcal{A}_w^-$, then $\sigma(\lambda - K) \subset \{0, \lambda\}$ (for every $\mathcal{X}$ as well).

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Recall that a point $\lambda \in \sigma(T)$ is a *normal eigenvalue* of T if λ is isolated in $\sigma(T)$ and $\mathcal{X} = \mathcal{X}_\lambda \oplus \mathcal{X}_\lambda'$, where $\mathcal{X}_\lambda$, $\mathcal{X}_\lambda'$ are the *spectral invariant subspaces* of T such that $\sigma(T_\lambda) = \{\lambda\}$ ($T_\lambda = T|\mathcal{X}_\lambda$ is the *restriction* of T to $\mathcal{X}_\lambda$) and $\lambda \notin \sigma(T|\mathcal{X}_\lambda')$, and $\mathcal{X}_\lambda$ is finite dimensional. Then we have

THEOREM 1. *$\mathcal{Q}_w^-$ is disjoint from the open dense set $\mathcal{B} = \{T \in \mathcal{L}(\mathcal{X})$: $\sigma_0(T) \cap [\mathbf{C}\setminus\{0\}] \neq \emptyset\}$, where $\sigma_0(T)$ denotes the set of all normal eigenvalues of T.*

In particular, if $B = \lambda - K \in \mathcal{Q}_w^- \cap (T)$, then either K is a compact quasinilpotent operator or $\sigma(K) = \{0, \lambda\}$. $\mathcal{Q}_w^- \cap (T)$ is nowhere dense in (T).

PROOF. That $\mathcal{B}$ is open and dense and, similarly, that $\mathcal{B} \cap (T)$ is open and dense in (T) follows from [**13**]. Thus, in order to complete the proof, it suffices to show that $\mathcal{Q}_w \cap \mathcal{B} = \emptyset$, and this follows from [**16**, Theorem 3]. (Indeed, the assumption that $\mathcal{H}$ is a Hilbert space is irrelevant in [**16**]: no modifications are necessary for the case of an arbitrary Banach space $\mathcal{X}$.) □

For a Hilbert space $\mathcal{H}$, Theorem 1 gives the best possible answer. Recall that the weak* (or ultraweak) topology of $\mathcal{L}(\mathcal{X})$ (i.e., the weak* topology as the dual of the Banach space of all trace class operators; see [**5**, p. 39]) is strictly stronger than the weak operator topology. Hence,

$$\mathcal{Q}_{w^*} = \bigcup \left\{ \operatorname{Ran}(\delta_A)^{-w^*} \cap \{A\}' \colon A \in \mathcal{L}(\mathcal{H}) \right\}$$

is a subset of $\mathcal{Q}_w$ and therefore, by Theorem 1,

$$(\mathcal{Q}_{w^*})^- \subset \mathcal{Q}_w^- \subset \mathcal{L}(\mathcal{H}) \setminus \mathcal{B}.$$

These inclusions are actually equalities; in fact, we have

THEOREM 2. *For an infinite dimensional Hilbert space $\mathcal{H}$, $\mathcal{L}(\mathcal{H})$ is the disjoint union of $\mathcal{B}$ and $(\mathcal{Q}_{w^*})^-$, i.e., $(\mathcal{Q}_{w^*})^- = \mathcal{Q}_w^- = \mathcal{L}(\mathcal{H}) \setminus \mathcal{B}$.*

The proof of this theorem will be given in §3. The author wishes to thank Professor L. A. Fialkow for providing useful references.

2. Complementary results.

LEMMA 1. (i) $\mathcal{Q}(\mathbf{C}^n) = \mathcal{Q}(\mathbf{C}^n)^- = \{Q \in \mathcal{L}(\mathbf{C}^n)$: *$Q$ is nilpotent*$\}$, $n = 1, 2, 3, \ldots$.

(ii) *For every $\mathcal{X}$, $\mathcal{Q}$ contains every finite rank nilpotent operator.*

PROOF. Let $Q_k, H_k \in \mathcal{L}(\mathbf{C}^k)$ be the operators defined by $Q_k e_1 = 0$, $Q_k e_j = e_{j-1}$, $j = 2, 3, \ldots, k$, and $H_k e_j = j e_j$, $j = 1, 2, 3, \ldots, k$, where $\{e_j\}_{j=1}^k$ is the canonical basis of $\mathbf{C}^k$. Then (as in [**6**, Chapter 19])

$$Q_k H_k - H_k Q_k = Q_k \in \operatorname{Ran}(\delta_{Q_k}) \cap \{Q_k\}'.$$

By considering finite direct sums and Jordan forms, we obtain (i); (ii) is a trivial consequence of (i) and the Hahn-Banach theorem. □

LEMMA 2. *Let* $T \in \mathcal{L}(\mathcal{X})$ *and assume that there exists an idempotent* P *such that* $PA = AP$ *for all* $A \in \{T\}'$. *Then* $T \in \mathcal{A}(\mathcal{X})$ *if and only if* $T|\text{Ran } P \in \mathcal{A}(\text{Ran } P)$ *and* $T|\text{Ker } P \in \mathcal{A}(\text{Ker } P)$.

In particular, if $\sigma(T)$ *is disconnected, then* $T \in \mathcal{A}(\mathcal{X})$ *if and only if* $T|\mathcal{X}_\sigma \in \mathcal{A}(\mathcal{X}_\sigma)$ *for every spectral invariant subspace* $\mathcal{X}_\sigma$ *(associated with a clopen subset* σ *of* $\sigma(T)$ *via Riesz decomposition theorem).*

The proof of the first part is straightforward. For the second part it is enough to recall that, if P_σ is the projection of $\mathcal{X}$ onto $\mathcal{X}_\sigma$ along its complementary spectral invariant subspace, then P_σ commutes with every $A \in \{T\}'$, so that $\{T\}'$ "splits", etc. (see, e.g., **[14**, §5]).

The following result is a minor improvement of the Theorem in **[15]**.

LEMMA 3. *If* T *is bounded below by a positive constant* α *and*

$$\phi(A, n) = (1/n) \sum_{i=0}^{n-1} \left(\|A^{n-i-1}\| \cdot \|A^i\| / \|A^{n-1}\|\right) \leqslant C(A),$$

where $C(A)$ *is a constant depending on* A, *for every* $A \in \{T\}'$ *such that* $\sigma(A) \subset D(2, 1) = \{\lambda : |\lambda - 2| < 1\}$, *then* $T \notin \mathcal{A}$.

PROOF. Assume that $T \in \mathcal{A}$; then

$$\lim(j \to \infty)\|\delta_A(X_j) - T\| = 0$$

for some $A \in \{T\}'$ and a suitable sequence $\{X_j\}_{j=1}^{\infty} \subset \mathcal{L}(\mathcal{X})$.

Replacing, if necessary, A by $2 + cA$ for some $c > 0$ small enough, we can directly assume that $\sigma(A) \subset D(2, 1)$ **[6**, Chapter 19]; then a trivial change of the proof of Theorem in **[15]** (see also **[6**, Problem 185]) shows that

$$\begin{aligned}\alpha &\leqslant (1/n)\left\{2\|A\| \cdot \|X_j\| + \|\delta_A(X_j) - T\| n\phi(A, n)\right\} \\ &\leqslant \left(2\|A\| \cdot \|X_j\|\right)/n + C(A)\|\delta_A(X_j) - T\|\end{aligned}$$

for all $n, j \geqslant 1$ and, a fortiori,

$$0 < \alpha \leqslant C(A)\|\delta_A(X_j) - T\| \to 0 \quad (j \to \infty),$$

a contradiction. □

REMARK. If $A \in \mathcal{L}(\mathbf{C}^k)$ and $\sigma(A) \subset D(2, 1)$, then $A = WJW^{-1}$ for some invertible W and some Jordan form J whose eigenvalues lie outside the closed unit disk $D^- = \{\lambda : |\lambda| \leqslant 1\}$; we obtain

$$\text{sp}(A)^i \leqslant \|A^i\| \leqslant \text{sp}(A)^i + k\|W\| \cdot \|W^{-1}\| \text{sp}(A) \leqslant \text{sp}(A)^i\left(1 + k\|W\| \cdot \|W^{-1}\|\right),$$

whence it readily follows that $\phi(A, n) = C(A) = (1 + k\|W\| \cdot \|W^{-1}\|)^2$.

3. The Hilbert space case.

PROOF OF THEOREM 2. Assume that $T \notin \mathcal{B}$; then T is similar to $T_0 \oplus T_1$, where T_0 is a nilpotent operator acting on a subspace $\mathcal{H}_0$ of finite dimension d, $0 \leqslant d < \infty$, and T_1 acts on an infinite dimensional subspace $\mathcal{H}_1$ and $\sigma_0(T_1) = \varnothing$.

Assume that $\mathcal{H}$ is *separable*; then it follows from the proof of [**9**, Theorem 3] that T_1 is the norm limit of a sequence $\{A_n\}_{n=1}^{\infty}$ in $\mathcal{L}(\mathcal{H}_1)$ such that $\{A_n\}' = \{A_n\}''$ (= double commutant) does not contain any nonzero compact operator.

According to [**19**, Corollary 1] $\text{Ran}(\delta_A)^{-w^*} = \mathcal{L}(\mathcal{H})$ if and only if $\{A\}'$ does not contain any nonzero trace class operator. Hence, $A_n \in \{A_n\}' \subset \text{Ran}(\delta_{A_n})^{-w^*} = \mathcal{L}(\mathcal{H}_1)$ for all $n = 1, 2, \ldots,$ and, a fortiori, $T_1 \in \mathcal{Q}_{w^*}(\mathcal{H}_1)^-$.

By using Lemma 1(i) and the fact that $\mathcal{Q}$ (and therefore $\mathcal{Q}^-$ too [**11**, §2]) is invariant under similarities, we conclude that T belongs to $\mathcal{Q}_{w^*}(\mathcal{H})^-$.

In the nonseparable case, $T_1 = \bigoplus_{\nu\in\Gamma} T_{1\nu}$ (T_1 is unitarily equivalent to the *orthogonal* direct sum of the family $\{T_{1\nu}\}_{\nu\in\Gamma}$), where $\text{card}(\Gamma) = \dim \mathcal{H}$ and $\sigma_0(T_{1\nu}) = \varnothing$ for all $\nu \in \Gamma$ (see, e.g., [**10**, Theorem 3]).

Combining this decomposition with the *proof* of [**9**, Theorem 3] we can find a sequence $\{A_n = \bigoplus_{\nu\in\Gamma} A_{1\nu n}\}_{n=1}^{\infty}$ (the decomposition of $\mathcal{H}_1$ being the same as for T_1, of course) of operators in $\mathcal{L}(\mathcal{H}_1)$ such that $A_{1\nu n}$ is similar to an operator in a finite family $\{B_1, B_2, \ldots, B_{r(n)}\}$, $\{B_j\}' = \{B_j\}''$ does not contain any nonzero compact operator and $\{A_n\}' \cap \mathcal{K}(\mathcal{H}_1) = \{0\}$ for every $n = 1, 2, 3, \ldots$. Exactly as in the separable case, we conclude that $T \in \mathcal{Q}_{w^*}(\mathcal{H})^-$. □

We close the paper with some remarks and open problems.

(a) If $\mathcal{H}$ is a complex separable infinite dimensional Hilbert space and K is unitarily equivalent to the orthogonal direct sum of the compact operators $\{(1/k)Q_k\}_{k=1}^{\infty}$ ($K \cong \bigoplus_{k=1}^{\infty}(1/k)Q_k$), where Q_k is the operator defined in the proof of Lemma 1(ii), then $K \in \mathcal{K}$ and the arguments used in that lemma show that $K \in \text{Ran}(\delta_K)^- \cap \{K\}'$; furthermore, if $L \cong \bigoplus_{k=1}^{\infty}(1/k!)^2 Q_k$, then $L \in \delta_K(\mathcal{K}) \cap \{K\}'$. Thus, $\mathcal{Q}\mathcal{K} = \bigcup\{\text{Ran}(\delta_K)^- \cap \{K\}' \colon K \in \mathcal{K}\}$ contains a quasinilpotent operator which is not nilpotent (similar examples can be obtained for an arbitrary infinite dimensional Banach space $\mathcal{X}$ by using Markushevich bases [**17**]).

Combining Lemma 1(ii) with a result of R. G. Douglas (every compact quasinilpotent is a norm limit of finite rank nilpotents [**7**, Problem 7]), it readily follows that $\mathcal{Q}^- \cap \mathcal{K} = \mathcal{Q}_w^- \cap \mathcal{K} = \mathcal{Q}\mathcal{K}^- = \{K \in \mathcal{K}\colon K \text{ is a quasi-nilpotent}\}$.

QUESTION 1. Is $\mathcal{Q} \cap \mathcal{K}$ a proper subset of $\mathcal{Q}\mathcal{K}^-$? Is $\mathcal{Q}\mathcal{K}$ a proper subset of $\mathcal{Q}\mathcal{K}^-$?

QUESTION 2. Does $1 - K \in \mathcal{Q}$ for some nonzero compact quasinilpotent K?

(b) By a celebrated theorem of J. H. Anderson [**1**], it readily follows that if N = (Nilpotent acting on a finite dimensional space) $\oplus\{\bigoplus_{\lambda\in\Lambda} I_\lambda\}$, where Λ is an arbitrary bounded subset of $\mathbf{C}$ and I_λ is the identity operator on a Hilbert space $\mathcal{H}_\lambda$ of *infinite dimension*, then $N \in \mathcal{Q}$. On the other hand, if

$$L = \int_{\sigma(T)} \lambda \, dE_\lambda$$

is the spectral decomposition of the normal operator L and there exists a Borel set $\Omega \subset \mathbf{C}\setminus\{0\}$ such that $E(\Omega) \neq 0$ and $L_\Omega = L|E(\Omega)\mathcal{H}$ has uniform finite multiplicity (the reader is referred to **[8]** for definitions and properties), then $L \notin \mathcal{Q}$. (Proof. $C(A_\Omega) =$ [multiplicity of L_Ω] for all $A_\Omega \in \{L_\Omega\}'$. Now apply Lemmas 2 and 3.) In particular, if U is the bilateral shift, then $U^{(n)} = U + U + \cdots + U$ (n copies) belongs to $\mathcal{Q}^- \setminus \mathcal{Q}$, $n = 1, 2, \ldots$.

QUESTION 3. Which normal operators belong to $\mathcal{Q}$? Does $U^{(\infty)} = U \oplus U \oplus U \oplus \ldots$ belong to $\mathcal{Q}$?

(c) If $\mathcal{H}$ is *separable*, then $T \in \mathcal{L}(\mathcal{H})$ is *biquasitriangular* if $\operatorname{ind}(\lambda - T) = 0$ for all $\lambda \in \mathbf{C}$ such that $\lambda - T$ is a semi-Fredholm operator **[2]**; for a nonseparable $\mathcal{H}$, biquasitriangularity is defined in terms of the *weighted spectra* of T (see **[10]**). Let (BQT) denote the class of all biquasitriangular operators acting in $\mathcal{H}$.

The above mentioned consequence of Anderson's theorem and the fact that $\mathcal{Q}^-$ is invariant under similarities imply that $\mathcal{S}(N)^- \subset \mathcal{Q}^-$, where $\mathcal{S}(N) = \{WNW^{-1}: W$ is invertible in $\mathcal{L}(\mathcal{H})\}$, for every N as in (b), whence we obtain the following

COROLLARY 1. *$(BQT)\setminus\mathcal{B} \subset \mathcal{Q}^-$. In particular, $(T)\setminus\mathcal{B} \subset \mathcal{Q}^-$.*

PROOF. It follows from **[3]** and **[10]** (see also **[12]**, **[18]**) that $(\mathrm{BQT})\setminus\mathcal{B} = [\bigcup\{\mathcal{S}(N)^-: N \text{ as in } b)\}]^-$ (the outer closure is irrelevant in the separable case). Hence, $(\mathrm{BQT})\setminus\mathcal{B} \subset \mathcal{Q}^-$. □

CONJECTURE. $\mathcal{Q}^- = (\mathrm{BQT})\setminus\mathcal{B}$.

(d) Is $\mathcal{L}(\mathcal{H}) = \bigcup\{\operatorname{Ran}(\delta_A)^-: A \in \mathcal{L}(\mathcal{H})\}$ **[15]**? Observe that if P_0 is a rank one orthogonal projection, the proof of Lemma 3 shows that $I + P_0$ is *orthogonal* to $\operatorname{Ran}(\delta_A)$ for all $A \in \{P_0\}'$ in the sense that

$$\operatorname{dist}\{I + P_0, \operatorname{Ran}(\delta_A)\} = \|I + P_0\| = 2.$$

QUESTION 4. Does $I + P_0 \in \operatorname{Ran}(\delta_A)^-$ for some $A \in \mathcal{L}(\mathcal{H})$?

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