A NECESSARY AND SUFFICIENT CONDITION FOR DISFOCALITY

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ABSTRACT. A criterion for the disfocality of a linear differential operator is given. The condition is analogous to a disconjugacy criterion of Pólya (1922).

1. Introduction. Consider the nth order linear differential operator $L$ defined by

$$Ly = y^{(n)} + a_1(t)y^{(n-1)} + \cdots + a_n(t)y,$$

where $a_i$ are continuous real valued functions on an open interval. The operator $L$ is said to be disconjugate on an interval $I$ if the only solution of $Ly = 0$ having $n$ zeros or more in $I$ counting multiplicities is the zero solution. If the only solution of $Ly = 0$ satisfying $y^{(i-1)}(t_i) = 0$ for some $t_i \in I$, $i = 1, \ldots, n$, $t_1 < t_2 < \cdots < t_n$, is the zero solution, then $L$ is said to be right disfocal on $I$, and $L$ is left disfocal if $t_1 > t_2 > \cdots > t_n$, $t_i \in I$. $y^{(i-1)}(t_i) = 0$ and $Ly = 0$ imply $y = 0$ on $I$. Pólya [4] shows that a necessary and sufficient condition for disconjugacy of $L$ on a compact interval $I$ is the existence of solutions $u_1, \ldots, u_{n-1}$ of $Ly = 0$ such that

$$W(u_1, \ldots, u_{k-1}, u_k) > 0, \quad k = 1, \ldots, n-1,$$

on $I$, where $W$ is the Wronskian determinant. Since it is clear from Rolle’s Theorem that left or right disfocality is a stronger restriction on $L$ than disconjugacy, it is of interest to decide if some additional simple restrictions on $u_1, \ldots, u_{n-1}$ provide a necessary and sufficient condition for left or right disfocality. A related question is considered by Nehari [3] in the context of certain generalized differential equations. In the present context one of Nehari’s results (Theorem 5.1) states that, if $a_n(t) \neq 0$ for each $t$ in an open interval $I$, then the only solution of $y^{(n)} + a_n(t)y = 0$ such that $y^{(i-1)}(t_i) = 0$, $t_i \in I$, $i = 1, \ldots, n$ is the zero solution if and only if all principal minors of the Wronskian matrix associated with a certain fundamental solution set are positive on $I$. He further shows (Theorem 7.2) that no minor of this matrix can vanish on $I$. The emphasis in the present paper is different because of the restriction on the order of the points where the successive derivatives vanish.

To motivate the results of this paper, consider an operator $L$ of order 2. Pólya’s criterion in this case is that $L$ is disconjugate on a compact interval $I$.
if and only if there exists a solution \( u_1 \) of \( Ly = 0 \) such that \( u_1 > 0 \) on \( I \); it is not difficult to see that \( L \) is right disfocal on \( I \) if and only if there exists a solution \( u_1 \) such that \( u_1 > 0, u'_1 > 0 \) on \( I \). This paper provides a generalization of the foregoing statement to operators of any order.

Throughout the paper attention is restricted to questions of right disfocality and the term "disfocal" is taken to mean "right disfocal". Analogous statements about left disfocality are obtained by formally replacing \( t \) by \(-t\) throughout.

2. A disfocality criterion. In the following theorem \( W(u_1, \ldots, u_n) \) denotes the Wronskian determinant \( \det[u_j^{(-1)}] \), \( i, j = 1, \ldots, n \); in particular \( W(u_j) = u_j \).

**Theorem.** A sufficient condition for the disfocality of \( L \) on a compact interval \([a, b]\) is the existence of solutions \( u_1, \ldots, u_{n-1} \) of \( Ly = 0 \) such that

\[
W(u_j^{(-1)}, \ldots, u_j^{(-1)}) > 0 \quad (1)
\]

for \( j = 1, \ldots, n - k + 1 \) and \( k = 1, \ldots, n - 1 \) on \([a, b]\). A necessary condition is that solutions \( u_1, \ldots, u_n \) exist such that (1) holds on \([a, b]\) for \( j = 1, \ldots, n - k + 1 \) and \( k = 1, \ldots, n \).

Before proving the theorem a few preliminary results will be established. Let \( \Omega(u_1, \ldots, u_n)(t_1, \ldots, t_n) \) denote the determinant of the \( n \times n \) matrix the \( i \)th column of which is

\[
\text{col}[u_i(t_1), u'_i(t_2), \ldots, u_{i-1}^{(n-1)}(t_n)],
\]

where \( u_1, \ldots, u_n \) are functions for which all the derivatives involved exist. Clearly

\[
\Omega(u_1, \ldots, u_n)(t, \ldots, t) = W(u_1, \ldots, u_n)(t),
\]

where \( W \) is the Wronskian determinant. The symbol \( \Omega(u_1, \ldots, u_n)(m; a, b) \) denotes \( \Omega(u_1, \ldots, u_n)(t_1, \ldots, t_n) \) when \( t_1 = \cdots = t_m = a, t_{m+1} = \cdots = t_n = b \). Also \( \Omega_j(u_1, \ldots, u_n)(t_1, t) \) denotes the determinant of the \( n \times n \) matrix for which the \( i \)th column is

\[
\text{col}[u_i(t_1), u'_i(t), u_{i+1}^{(j+1)}(t), \ldots, u_i^{(j+n-2)}(t)],
\]

if \( n > 1 \), and \( \Omega_j(u_1)(t_1, t) = u_i(t_1) \). Let \( \beta(a) = \sup\{b\} \) such that \( b > a \) and \( L \) is disfocal on \([a, b]\) and let \( \alpha(b) = \inf\{a\} \) such that \( a < b \) and \( L \) is disfocal on \([a, b]\). It is clear that \( a < \beta(a) \) and \( \alpha(b) < b \) whenever \( \beta, \alpha \) exist since the existence of a nontrivial solution to

\[
Ly = 0, \quad y(t_1) = y'(t_2) = \cdots = y^{(n-1)}(t_n) \quad (3)
\]

is equivalent to \( \Omega(u_1, \ldots, u_n)(t_1, \ldots, t_n) = 0 \), where \( u_1, \ldots, u_n \) is a fundamental solution set; since \( W(u_1, \ldots, u_n) \neq 0 \) it follows by continuity that \( \Omega(u_1, \ldots, u_n)(t_1, \ldots, t_n) \neq 0 \) if \( t_1, \ldots, t_n \) are sufficiently close together.
PROPOSITION 1. (a) If \( b = \beta(a) \), then \( L \) is not disfocal on \([a, b]\) and, for each nontrivial solution \( y \) of (3) such that \( a < t_1 < \cdots < t_n < b \), the condition \( t_1 = a, t_n = b \) holds. If a solution \( y \) exists for such a set of points \( \{t_i\} \), then it is unique to within a constant multiple and \( t_i \) is the only zero in \([t_{i-1}, b]\) of \( y^{(i-1)} \), \( i = 1, \ldots, n \) \((t_0 = a)\).

(b) If \( a < t_i < \cdots < t_{j-1} < b \) for one of the solutions described in (a), then for every choice of the \((j-i)\)-tuple \( \{t_i, \ldots, t_{j-1}\} \) such that
\[
a = t_1 = \cdots = t_{i-1} < t_i < \cdots < t_{j-1} < t_j = \cdots = t_n = b
\]
there exists a nontrivial solution \( y \) of (3). In particular, there exist \( m, 1 < m < n \), and a nontrivial solution \( y \) to (3) with
\[
a = t_1 = \cdots = t_m < t_{m+1} = \cdots = t_n = b.
\]

PROOF. First \( L \) is not disfocal on \([a, b]\) since, for each \( c > b \), there exists a nontrivial solution \( v \) to (3) such that \( a < f_1 < \cdots < f_n < c \) and hence
\[
\int_ux(.,. ., \cdot) f(.,. ., \cdot) = 0
\]
where \( u_1, \ldots, u_n \) is a fundamental solution set. By continuity this also holds for \( c = b \). If \( \Omega(u_1, \ldots, u_n)(t_1, \ldots, t_n) = 0, a < t_1 < \cdots < t_n < b \), then \( t_n = b \) since \( t_n < b \) contradicts the extremality of \( b \). Also \( t_1 \) is the only zero of \( y^{(i-1)} \) in \([t_{i-1}, b]\) since otherwise the extremality of \( b \) is contradicted by Rolle's Theorem; similarly \( y \) is unique up to within a constant multiple since the existence of two or more linearly independent solutions implies the existence of a nontrivial solution to (3) with a zero in \((t_1, b)\). It will be shown that if any of the points \( t_m \) \((m < n)\) is in the open interval \((a, b)\) then the cofactors of the row \((u_1^{(i-1)}(t_m), \ldots, u_n^{(i-1)}(t_m))\) in the \(n \times n\) matrix defined by (2) are all zero. From this it follows that \( \Omega(u_1, \ldots, u_n)(t_1, \ldots, t_n) = 0 \) for all choices of \( t_m \), in particular for all \( t_m \) satisfying \( t_{m-1} < t_m < t_{m+1} \). This also shows that \( t_1 = a \) since, if \( t_1 \in (a, b) \), this principle implies that
\[
W(u_1, \ldots, u_n)(b) = \Omega(u_1, \ldots, u_n)(b, \ldots, b) = 0
\]
contradicting the fact that \( u_1, \ldots, u_n \) is a fundamental solution set. Thus all the statements in Proposition 1 are proved subject to the truth of the assertion concerning the cofactors of the row \((u_1^{(i-1)}(t_m), \ldots, u_n^{(i-1)}(t_m))\). To prove this assertion, suppose the cofactors are not all zero; suppose also that \( t_m < t_{m+1} \). Then the determinant of the matrix for which the \(i\)th column is
\[
\text{col}[u_j(t_1), \ldots, u_j^{(i-2)}(t_{i-1}), u_j(t_i), u_j^{(i)}(t_{m+1}), \ldots, u_j^{(n-1)}(t_n)]
\]
is a nontrivial solution \( y \) of (3), and also \( y^{(m)}(t_m) \neq 0 \) since, as before, \( y^{(m)}(t_m) = 0 \) contradicts the extremality of \( b \) by Rolle's Theorem. Next suppose that \( t_i = t_{i+1} = \cdots = t_m \) \((i < i+1 < \cdots < m)\) and \( t_j = t_{j+1} = \cdots = t_n = b \) \((m < j < j+1 < \cdots < n)\) and that all other points \( (\text{if any}) \) in the \(n\)-tuple \( \{t_1, \ldots, t_n\} \) are different from \( t_m \) and \( t_n \). Let
\[
Y(t, s) = \Omega(u_1, \ldots, u_n)(t_1, \ldots, t_{i-1}, t, \ldots, t_i, t_{m+1}, \ldots, t_{j-1}, s, \ldots, s);
\]
then \( Y(t_m, t_n) = 0 \) and \((\partial/\partial t) Y(t_m, t_n) = y^{(m)}(t_m) \neq 0 \) so that, by the Implicit
Function Theorem, there exists a continuous function $T$ defined on a neighbourhood of $t_n$ such that $T(t_n) = t_m$ and $Y(T(s), s) = 0$. In particular, if $s$ is any number in a sufficiently small neighbourhood of $t_n (= b)$, then $t_{i-1} < T(s) < t_{m+1}$ so that the extremality of $b$ is contradicted. Thus the hypothesis that, in the matrix defined by (2), the cofactors of the row $(u_1^{(m-1)}(t_m), \ldots, u_n^{(m-1)}(t_m))$ are not all zero is false.

**Proposition 2.** Suppose $b = \beta(a)$ and $m$ is minimal such that $\Omega(u_1, \ldots, u_n)(m; a, b) = 0$. Then $(\partial/\partial a)\Omega(u_1, \ldots, u_n)(m; a, b) \neq 0$ and there exists a continuously differentiable function $\alpha_m$ defined on a neighbourhood $U$ of $b$ such that $a = \alpha_m(b)$ and $\Omega(u_1, \ldots, u_n)(m; c, d) = 0$ if $c = \alpha_m(d)$; also if $d \in U$ and $d < b$ then $\alpha_m(d) < a$.

**Proof.** Under the conditions of this proposition the cofactors of the row $(u_1^{(m-1)}(a), \ldots, u_n^{(m-1)}(a))$ in the matrix for which the $i$th column is given by (2) with $a = t_1 = \cdots = t_m, t_{m+1} = \cdots = t_n = b$ are not all zero; otherwise $\Omega(u_1, \ldots, u_n)(m - 1; a, b) = 0$ contradicting the minimality of $m$. Next let $y(t)$ be the determinant of the matrix the $i$th column of which is given by (4) with $a = t_1 = \cdots = t_{m-1}, t_{m+1} = \cdots = t_n = b$. This function is a non-trivial solution of

$$Ly = 0, \quad y(a) = \cdots = y^{(m-1)}(a) = y^{(m)}(b) = \cdots = y^{(n-1)}(b) = 0$$

which, by Proposition 1, must satisfy $y^{(m)}(a) \neq 0$. Therefore $\Omega(u_1, \ldots, u_n)(m; a, b) = 0$ and

$$(\partial/\partial a)\Omega(u_1, \ldots, u_n)(m; a, b) = y^{(m)}(a) \neq 0.$$ 

By the Implicit Function Theorem the existence of $\alpha_m$ follows. The fact that $d < b$ implies $\alpha_m(d) < a$ follows from $b = \beta(a)$.

**Proposition 3.** Suppose that $\beta(a)$ exists. Then $b = \beta(a)$ implies $a = \alpha(b)$. The function $\beta$ is left-continuous and increasing but need not be continuous and $\alpha$ is continuous and nondecreasing on its domain.

**Proof.** From the definitions of $\alpha, \beta$ it follows that both functions are nondecreasing. Proposition 1 implies that $\beta$ is increasing and if $b = \beta(a)$, then $a = \alpha(b)$. From Proposition 2 it follows that $\beta(a - ) > \beta(a)$ and therefore, since $\beta$ is increasing, $\beta(a - ) = \beta(a)$. Also, since $\alpha$ is nondecreasing, $\beta(a - ) < v < \beta(a + )$ implies $a = \alpha(v)$ and since $\beta$ is increasing $\alpha$ is continuous. But it is not necessarily the case that $\beta(a + ) = \beta(a)$. Consider the second order operator

$$Ly = W(\phi_1, \phi_2, y)/W(\phi_1, \phi_2),$$

where $\phi_1, \phi_2$ are chosen so that

$$\phi_1(2) = 0, \quad \phi_1(t) < 0, \quad t \in (0, 3),$$

$$\phi_2(0) = \phi_2(1) = \phi_2(2) = 0, \quad \phi_2(t) > 0, \quad t \in (0, 1) \cup (1, 2),$$

$$\phi_2(t) < 0, \quad t \in (2, 3).$$
Thus $L$ is well defined since $W(\phi_1, \phi_2) \neq 0$, and $\beta(0) = 1$ because $\phi_2(0) = \phi_2(1) = 0$ and $\phi_2(t) > 0$, $t \in (0, 1)$; $\beta(0+) = 2$ since, if $L = 0$ and $y(t_0) = 0$, $t_0 \in (0, 2)$, then $y = c_1\phi_1 + c_2\phi_2$ where $c_1c_2 < 0$ so that if $t_0$ is sufficiently close to $0$, the first zero of $y'$ is greater than and close to $2$.

**PROPOSITION 4.** The operator $L$ is disfocal on $(c, d]$ if and only if the only solution of (3) such that $c < t_1 < t_2 < \cdots < t_n < d$ is the zero solution.

**Proof.** The necessity of this condition is obvious. To see its sufficiency observe that if $L$ is not disfocal on $(c, d]$, then there is a subinterval $[a, \beta(a)]$ of $(c, d]$ and, by Propositions 1 and 2, a natural number $m$ $(1 < m < n)$ such that $\Omega(u_1, \ldots, u_n)(m; t, \beta(a))$ changes sign at $t = a$. But this cannot occur since, if $W(u_1, \ldots, u_n) > 0$, then $\Omega(u_1, \ldots, u_n)(t_1, \ldots, t_n) > 0$ if $c < t_1 < t_2 < \cdots < t_n < d$, and hence $\Omega(u_1, \ldots, u_n)(t_1, \ldots, t_n) > 0$ if $c < t_1 < t_2 < \cdots < t_n < d$; in particular $\Omega(u_1, \ldots, u_n)(m; t, \beta(a)) > 0$ if $c < t < \beta(a)$.

The proof of the theorem will require the use of the following determinantal identities. If $A_1, B_1, A, B, C, D$ are any real numbers, then

$$
\begin{vmatrix}
A_1 & B_1 \\
A & B
\end{vmatrix}
= A_1 \begin{vmatrix}
A & B \\
C & D
\end{vmatrix}
+ C \begin{vmatrix}
A & B_1 \\
A & B
\end{vmatrix}.
$$

(5)

Let $a_{r_1, \ldots, r_m}^{s_1, \ldots, s_m}$ denote the minor of the $n \times n$ matrix $[a_{ij}]$ determined by the rows $r_1, \ldots, r_m$ and the columns $s_1, \ldots, s_m$ and let $b'_j = a_{12, \ldots, p, p+1, \ldots, n}^{1, 2, \ldots, p, p+1, \ldots, n}$; then

$$
a_{12, \ldots, n}^{12, \ldots, p} = b_{12, \ldots, n}^{12, \ldots, p}, \quad p = 1, \ldots, n - 1.
$$

(6)

The relation (6) is known as Sylvester's Identity; cf. [1, p. 32]. It should be observed that, by row and column interchanges, any $p \times p$ minor may play the role of $a_{12, \ldots, p}$.

**Lemma 1.** Suppose $u_1, \ldots, u_n$ are such that (1) holds on $[a, b]$ for $j = 1$ and $k = 1, \ldots, n$. Then

$$
\Omega_0(u_1, \ldots, u_k)(t, t) > 0, \quad k = 1, \ldots, n,
$$

if $a < t < b$, and equality holds only if $k > 1$ and $t = t_1$.

**Proof.** This is a special case of Theorem V of Pólya [4]. Pólya outlines his proof; a detailed proof may be found in [2, pp. 376–378].

**Lemma 2.** The identity

$$
\Omega_0(u_1, \ldots, u_k)(t, t) W(u_1^{-1}, \ldots, u_k^{-1})(t)
= \Omega_0(u_1, \ldots, u_{k-1})(t, t) W(u_1^{-1}, \ldots, u_{k-1}^{-1})(t)
+ \Omega_{j-1}(u_1, \ldots, u_k)(t, t) W(u_j^0, \ldots, u_{k-1}^0)(t)
$$

holds for all functions $u_1, \ldots, u_k$ for which these expressions exist.

**Proof.** From (6) with $n = k, p = k - 2$ and

$$
a_{12, \ldots, k}^{12, \ldots, k-2} = \Omega_0(u_1, \ldots, u_k)(t, t), \quad a_{12, \ldots, k-2}^{12, \ldots, k-2} = W(u_j^0, \ldots, u_{k-1}^0)(t),
$$
it follows that
\[ \Omega_j(u_1, \ldots, u_k)(t_1, t) W(u_1^{(j)}, \ldots, u_{k-2}^{(j)})(t) = \begin{vmatrix} A_1 & B_1 \\ C & D \end{vmatrix} \]
where
\[ A_1 = \Omega_j(u_1, \ldots, u_{k-1})(t_1, t), \quad B_1 = \Omega_j(u_1, \ldots, u_{k-2}, u_k)(t_1, t), \]
\[ C = W(u_1^{(j)}, \ldots, u_{k-1}^{(j)})(t), \quad D = W(u_1^{(j)}, \ldots, u_{k-2}^{(j)}, u_k^{(j)})(t). \]

Now Lemma 2 follows from (5) with
\[ A = W(u_1^{(k-1)}, \ldots, u_{k-1}^{(k-1)})(t), \quad B = W(u_1^{(k-1)}, \ldots, u_{k-2}^{(k-1)}, u_k^{(k-1)})(t), \]
since, again by Sylvester's Identity (6),
\[ \begin{vmatrix} A & B \\ C & D \end{vmatrix} = \begin{vmatrix} A & B \\ C & D \end{vmatrix}, \]
\[ \Omega_{j-1}(u_1, \ldots, u_k)(t_1, t) W(u_1^{(j)}, \ldots, u_{k-2}^{(j)})(t) = \begin{vmatrix} A_1 & B_1 \\ A & B \end{vmatrix}. \]

**Proof of the Theorem.** To prove the necessity of the condition given suppose that \( L \) is disfocal on \([a, b]\). Then \( L \) is disfocal on \([\tau, b]\) for some \( \tau < a \), by Proposition 3. Now let \( u_1, \ldots, u_n \) be such that \( Lu_i = 0 \) and
\[ u_j^{(j-1)}(\tau) = 0, \quad j = 1, \ldots, n - i, \]
\[ (-1)^{j-1} u_i^{(n-i-1)}(\tau) > 0, \quad i = 1, \ldots, n. \] (7)
Then \( W(u_1^{(j-1)}, \ldots, u_k^{(j-1)}) \neq 0 \) on \((\tau, b]\) for \( j = 1, \ldots, n - k + 1 \) and \( k = 1, \ldots, n \) since, if \( W(u_1^{(j-1)}, \ldots, u_k^{(j-1)})(\sigma) = 0 \) for some \( \sigma \in (\tau, b] \), there exists a nontrivial solution \( y = c_1 u_1 + \cdots + c_k u_k \) of \( Ly = 0 \) such that
\[ y(\tau) = y'(\tau) = \cdots = y^{(n-k-1)}(\tau) = 0, \]
\[ y^{(j-1)}(\sigma) = y^{(j)}(\sigma) = \cdots = y^{(j+k-2)}(\sigma) = 0, \]
and by Rolle's Theorem there exist \( t_1, \ldots, t_n, \tau < t_1 < t_2 < \cdots < t_n < \sigma \), such that \( y^{(j-1)}(t_j) = 0, \quad j = 1, \ldots, n \), contradicting the disfocality of \( L \) on \([\tau, b]\). It remains to show that the Wronskians \( W(u_1^{(j-1)}, \ldots, u_k^{(j-1)}) \) are all positive on \((\tau, b]\) and hence on \([a, b]\). Condition (7) implies that, near \( t = \tau \),
\[ u_j^{(j-1)}(t) = [1 + o(1)] v_j^{(j-1)}(t), \quad j = 1, \ldots, n - i + 1, i = 1, \ldots, n, \]
where \( v_j(t) = \mu_j(t - \tau)^{n-i}, \quad (-1)^{j-1} \mu_j > 0 \). Therefore the Wronskians \( W(u_1^{(j-1)}, \ldots, u_k^{(j-1)}) \) are positive near \( \tau \) and thus throughout \((\tau, b]\) for \( j = 1, \ldots, n - k + 1, k = 1, \ldots, n \).

The sufficiency of the condition given in the Theorem will first be proved in the case that it holds on the closed interval \([a, b]\). Choose a solution \( u_n \) so that \( W(u_1, \ldots, u_n) > 0 \) on \([a, b]\), i.e. (1) holds for \( k = n \) also. Under these circumstances it will be seen that
\[ \Omega_j(u_1, \ldots, u_k)(t_1, t) > 0 \] (8)
for \( a < t_1 < t < b, k = 1, \ldots, n, j = 1, \ldots, n - k \). It is obvious in the case
$k = 1, j = 1, \ldots, n - 1$; also, from Lemma 1, $\Omega_0(u_1, \ldots, u_k)(t_1, t) > 0, a < t_1 < t < b, k = 1, \ldots, n$. Thus, from Lemma 2, if

$$\Omega_j(u_1, \ldots, u_{k-1})(t_1, t) > 0, \quad \text{and} \quad \Omega_{j-1}(u_1, \ldots, u_k)(t_1, t) > 0,$$

then $\Omega_j(u_1, \ldots, u_k)(t_1, t) > 0, a < t_1 < t < b$, so that, by induction on $j$ and $k$, (8) holds as stated. It will now be shown by induction on $n$ that

$$\Omega(u_1, \ldots, u_n)(t_1, \ldots, t_n) > 0, \quad a < t_1 < \cdots < t_n < b. \quad (9)$$

This is trivial in the case $n = 1$; assume that it is true for operators of order less than $n$. Consider the functions $v_1, \ldots, v_{n-1}$ defined by

$$v_i(t) = \Omega_i(u_1, u_{i+1})(t_1, t), \quad a < t_1 < t < b.$$

Since, from Sylvester’s Identity (6) with $n = k + 1, p = 1$,

$$W(v^{(i-1)}, \ldots, v^{(k-1)})(t) = \Omega_j(u_1, \ldots, u_{k+1})(t_1, t)[u_i(t_1)]^{k-1},$$

it follows from (8) that

$$W(v^{(i-1)}, \ldots, v^{(k-1)}) > 0 \quad \text{on} \quad [t_1, b], j = 1, \ldots, n - k, k = 1, \ldots, n - 1,$$

and hence, by the induction hypothesis, that

$$\Omega(v_1, \ldots, v_{n-1})(t_2, \ldots, t_n) > 0, \quad \text{if} \quad t_1 < t_2 < \cdots < t_n < b. \quad (10)$$

But, again by (6),

$$\Omega(v_1, \ldots, v_{n-1})(t_2, \ldots, t_n) = \Omega(u_1, \ldots, u_n)(t_1, \ldots, t_n)[u_i(t_1)]^{n-2}$$

and therefore from (10) the inequality (9) holds, so that the sufficiency assertion of the theorem is proved for operators of order $n$ when (1) holds on $[a, b]$.

When (1) holds on $[a, b]$ it also holds, by continuity on $[\tau, b)$, for some $\tau < a$ and hence $\beta(\tau) > b$ from what has been already proved. Therefore $\beta(a) > b$, by Proposition 3, and $L$ is disfocal on $[a, b]$ in this case also.

**Example.** It can be seen that the operator $D^n$ is disfocal on every interval $[a, b]$ since the functions

$$u_i(t) = (-1)^{i-1}(t - \tau)^{n-i}, \quad i = 1, \ldots, n,$$

satisfy (1) for $j = 1, \ldots, n - k + 1$ and $k = 1, \ldots, n$ on $[a, b]$ if $\tau < a$.

**References**


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