A MAXIMUM PRINCIPLE
FOR SEMILINEAR PARABOLIC SYSTEMS

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Abstract. We develop a criterion insuring that every component of the solution to a system of semilinear parabolic equations is strictly positive for positive time. This criterion involves the strict (component-wise) positiveness of solutions to a related ordinary differentiable system.

In this note we present a result concerning the strict positiveness of solutions $\tilde{u} = (u_1, \ldots, u_m)$ to the following system of weakly coupled parabolic equations

$$\frac{\partial}{\partial t} u_k(t, x) = L_k u_k(t, x) + F_k(t, x, \tilde{u}(t, x), \Delta u_k(t, x)), \quad t > 0, \ x \in \Omega, \ y \in \partial \Omega, \ k = 1, \ldots, m,$$

$$B_k u_k(t, y) = 0, \quad u_k(0, x) = \chi_k(x) > 0. \quad (PS)$$

Here $\Omega \subset \mathbb{R}^n$ ($n > 1$) is a bounded domain with smooth boundary, $\partial \Omega$, and $\Delta$ is the gradient operator (with respect to $x \in \mathbb{R}^n$). Also, for each $k \in \{1, \ldots, m\}$, $L_k$ is a uniformly elliptic operator with the representation

$$L_k \sim \sum_{i,j=1}^n \frac{\partial}{\partial x_i} \left( a_{k,ij}(x) \frac{\partial}{\partial x_j} \right)$$

with real-valued, smooth coefficient functions $a_{k,ij} = a_{k,ji}$, and $B_k$ is a boundary operator on $(0, \infty) \times \partial \Omega$ of the form

$$B_k u_k(t, y) = b_k(y) u_k(t, y) + \delta_k \frac{\partial}{\partial y} u_k(t, y)$$

where $\nu$ is the outward normal on $\partial \Omega$ and either $\delta_k = 0$ and $b_k(y) \equiv 1$ on $\partial \Omega$, or $\delta_k = 1$ and $b_k(y) > 0$ on $\partial \Omega$. Moreover, the real-valued function $F_k$ is $C^2$ on $[0, \infty) \times \Omega \times \mathbb{R}^m \times \mathbb{R}^n$ and for each $R > 0$ there are numbers $M > 0$ and $\gamma \in [0, 2)$ such that

$$|F_k(t, x, \xi, \eta)| \leqslant M \left( 1 + |\eta|^\gamma \right)$$

whenever $t \in [0, R], \ x \in \bar{\Omega}, \ \eta \in \mathbb{R}^n$, and $\xi \in \mathbb{R}^m$ with $||\xi|| \leqslant R$.

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The purpose of this note is to obtain a criterion insuring the strict positiveness of every component for solutions to (PS). Our criterion is based on the behavior of solutions to a related system of ordinary differential equations in \( \mathbb{R}^m \). Let \( \theta \) denote the zero vector in \( \mathbb{R}^m \) and for each \( \xi = (\xi_i)_1^m \) and \( \eta = (\eta_i)_1^m \) in \( \mathbb{R}^m \) write \( \xi > \eta \) only in case \( \xi_i > \eta_i \) for \( i = 1, \ldots, m \), and write \( \xi \gg \eta \) only in case \( \xi_i > \eta_i \) for \( i = 1, \ldots, m \). Also, let \( \theta \) denote the zero of \( \mathbb{R}^m \). It is assumed that \( F = (F_k)_1^m \) satisfies the following type of quasipositive condition:

\[
\text{if } k \in \{1, \ldots, m\} \text{ and } \xi \in \mathbb{R}^m \text{ with } \xi > \theta \text{ and } \xi_k = 0, \text{ then } F_k(t, x, \xi, \theta) > 0 \text{ for all } (t, x) \in [0, \infty) \times \Omega. \tag{1}
\]

Throughout this note it is assumed that \( x_0 \) is a (fixed) member of \( \Omega \) and that \( g = (g_k)_1^m \) is defined on \( [0, \infty) \times \mathbb{R}^m \) by

\[
g_k(t, \xi) = F_k(t, x_0, \xi, \theta) \text{ for } (t, \xi) \in [0, \infty) \times \mathbb{R}^m \text{ and } k = 1, \ldots, m. \tag{2}
\]

Our comparison system of ordinary differential equations is

\[
z'(t) = g(t, z(t)), \quad t > 0. \tag{ODE}
\]

From (1) we see that if \( k \in \{1, \ldots, m\} \) and \( \xi > \theta \) with \( \xi_k = 0 \), then \( g_k(t, \xi) > \theta \), and therefore it is easily deduced that if \( z \) is a solution to (ODE) with \( z(t_0) > \theta \), then \( z(t) > \theta \) for all \( t > t_0 \). Our second main supposition is

\( \Gamma \) is a subset of \( \{1, \ldots, m\} \) with the property that for each solution \( z = (z_i)_1^m \) to (ODE), the conditions \( z(t_0) > \theta \) and \( z_i(t_0) > 0 \) for \( i \in \Gamma \) imply \( z(t) \gg \theta \) for all \( t > t_0 \).

One should note that if \( z(t_0) \gg \theta \), then \( z(t) \gg \theta \) for all \( t > t_0 \), and hence (3) is always satisfied with \( \Gamma = \{1, \ldots, m\} \). If each solution \( z \) has the property that \( z(t_0) > \theta \) implies \( z(t) \gg \theta \) for all \( t > t_0 \), then (3) holds with \( \Gamma \) the empty set.

The final assumption is technical, but is less restrictive than requiring \( F \) to be quasimonotone:

\[
\text{if } t > 0, k \in \{1, \ldots, m\} - \Gamma, \text{ and } \xi > \theta \text{ is such that } \xi_k = 0 \text{ and } F_k(t, x_0, \xi, \theta) = 0, \text{ then } F_k(t, x_0, \eta, \theta) = 0 \text{ for all } \theta < \eta < \xi. \tag{4}
\]

Observe that (4) is a consequence of (1) whenever

\[
\frac{\partial}{\partial \xi_j} F_i(t, x_0, \xi, \theta) > 0 \quad \text{for } i \neq j \text{ (i.e. whenever } F \text{ is quasimonotone).}
\]

**Theorem.** Suppose \( \Gamma \) is a subset of \( \{1, \ldots, m\} \) and (1), (3) and (4) are satisfied. Suppose also that the nonnegative initial function \( \chi = (\chi_k)_1^m \) is continuous on \( \bar{\Omega} \) and that \( \chi_k \) is nontrivial for each \( k \in \Gamma \). Then there is a \( T > 0 \) such that the solution \( \bar{u} = (u_k)_1^m \) to (PS) exists on \( [0, T) \times \bar{\Omega} \) and satisfies \( \bar{u}(t, x) \gg \theta \) for all \( (t, x) \in (0, T) \times \bar{\Omega} \).

For our proof we use the following result:

**Lemma.** For each continuous, nonnegative \( \chi = (\chi_k)_1^m \) on \( \bar{\Omega} \) there is a \( T > 0 \) such that (PS) has a solution \( \bar{u} = (u_k)_1^m \) on \( [0, T) \times \bar{\Omega} \) with the property that if
Proof of Theorem. Let $\vec{u} = (u_k)_{i=1}^{m}$ be the solution to (PS) guaranteed by the Lemma and suppose, for contradiction, that $u_k(t, x) = 0$ for some $(t_i, x_i) \in (0, T) \times \Omega$ and some $k$. Using the Lemma again, we see that if $\Gamma_0 = \{k: u_k \equiv 0 \text{ on } (0, T) \times \Omega\}$ and $\Gamma_1 = \{k: u_k > 0 \text{ on } (0, T) \times \Omega\}$ then $\Gamma_0 \neq \emptyset$, $\Gamma_0 \cup \Gamma_1 = \{1, \ldots, m\}$, $\Gamma_1 \supset \Gamma$, and $\Gamma_0 \cap \Gamma_1 = \emptyset$. It is immediate from (PS) that

$$F_k(t, x, \vec{u}(t, x), \theta) \equiv 0 \text{ on } (0, T) \times \Omega \text{ and } k \in \Gamma_0. \quad (5)$$

Choose $0 < a < b < T$ and select $\xi = (\xi_i)_{i=1}^{m} \in \mathbb{R}^m$ such that $\xi_i = 0$ for $i \in \Gamma_0$ and $0 < 2\xi_i < u_i(t, x_0)$ for $i \in \Gamma_1$ and $t \in [a, b]$. From (4) and (5) we have

$$F_k(t, x_0, \eta, \theta) = 0 \quad \text{for } t \in [a, b], \theta < \eta < 2\xi_i, k \in \Gamma_0. \quad (6)$$

Now define the function $z = (z_i)_{i=1}^{m}$ on $[a, b]$ as follows: $z_i(t) \equiv 0$ on $[a, b]$ for $i \in \Gamma_0$ and $(z_i: i \in \Gamma_1)$ satisfies the initial value problem

$$z_i(t) = F_i(t, x_0, z(t), \theta), \quad t \in [a, b], z_i(a) = \xi_i, i \in \Gamma_1. \quad (7)$$

Since $\xi_i > 0$ for $i \in \Gamma_1$, choose $c \in (a, b)$ such that $0 < z_i(t) < 2\xi_i$ for all $t \in [a, c]$ and $i \in \Gamma_1$, and then note that

$$0 = F_i(t, x_0, z(t), \theta), \quad t \in [a, c], z_i(a) = \xi_i = 0, i \in \Gamma_0, \quad (7)'$$

by (6). From the definition of $g$ (see (2)) it is immediate from (7) and (7)' that $z$ is a solution to (ODE) on $[a, c]$ with $z_i(a) > 0$ for $i \in \Gamma_1 \supset \Gamma$ and $z_i(t) \equiv 0$ for $i \in \Gamma_0$. This contradicts assumption (3) and we conclude that $\Gamma_0$ must be empty. This proves the Theorem once the Lemma is established.

Proof of Lemma. The quasipositive assumption (1) along with a weak form of the maximum principle for parabolic equations implies that the solution $\vec{u} = (u_i)_{i=1}^{m}$ satisfies $\vec{u}(t, x) \geq \theta$ on $[0, T) \times \Omega$ for some $T > 0$ (see, e.g., the techniques in Amann [1], Lemmert [4], Lightbourne and Martin [5], and Volkmann [8]). Fix a number $k$ in $\{1, \ldots, m\}$ and for $(t, x) \in [0, T) \times \Omega$, $r \in \mathbb{R}$, and $l \in \{1, \ldots, n\}$ define

$$w_{k,t,x,r} = (\xi_k)_1 = u_i(t, x) \text{ for } i \neq k$$

and

$$q_{k,t,x,l,r} = (\eta_l)_1 \text{ where } \eta_l = r, \eta_j = 0 \text{ for } j < l \text{ and }$$

$$\eta_l = \partial / \partial x_j u_k(t, x) \text{ for } j > l.$$ 

Now define

$$\alpha_k(t, x) = u_k(t, x)^{-1} \int_0^t u_k(t, x)^{m} \partial / \partial x_k F_k(t, x, w_{k,t,x,r}, \Delta u_k(t, x)) \, dr$$
and
\[ \beta_{k,i}(t, x) = h_i(t, x)^{-1} \int_0^{h_i(t, x)} \frac{\partial}{\partial q_i} F_k(t, x, w_{k,t,x,0}, \theta) \, dq_i \]
where \( h_i(t, x) = \frac{\partial}{\partial x_i} u_k(t, x) \) \((i = 1, \ldots, n)\). Then
\[ F_k(t, x, u(t, x), \Delta u_k(t, x)) = F_k(t, x, w_{k,t,x,0}, \theta) + \alpha_k(t, x) u_k(t, x) + \sum_{i=1}^n \beta_{k,i}(t, x) \frac{\partial}{\partial x_i} u_k(t, x) \]
and since \( F_k(t, x, w_{k,t,x,0}, \theta) > 0 \) by (1) it follows from (PS) that
\[ \frac{\partial}{\partial t} u_k(t, x) > L_k u_k(t, x) + \alpha_k(t, x) u_k(t, x) + \sum_{i=1}^n \beta_{k,i}(t, x) \frac{\partial}{\partial x_i} u_k(t, x) \]
for all \((t, x) \in (0, T) \times \Omega\). Since \( u_k > 0 \) on \((0, T) \times \Omega\) we have from a strong form of the maximum principle [7, pp. 173 and 175] that \( T \) can be chosen so that either \( u_k \equiv 0 \) on \((0, T) \times \Omega\) or \( u_k > 0 \) on \((0, T) \times \Omega\). This completes the proof indication of the Lemma.

As an illustration of this result, we consider the mathematical model of a cellular control process with either positive or negative feedback. (See Griffith [2], [3].) This model is the system of three ordinary differential equations
\[ z_1' = -\alpha z_1 + h(z_3), \quad z_1(0) > 0, \]
\[ z_2' = -\beta z_2 + z_1, \quad z_2(0) > 0, \]
\[ z_3' = -\gamma z_3 + z_2, \quad z_3(0) > 0, \]
where \( \alpha, \beta, \gamma \) are positive constants and the function \( h \) is defined on \([0, \infty)\) by either \( h(r) = r^\sigma(1 + r^\sigma)^{-1} \) (positive feedback) or \( h(r) = (1 + r^\sigma)^{-1} \) (negative feedback), and \( \sigma > 1 \) is a constant. Defining \( F = (F_i)_{i=1}^3 \) by the right side of (8):
\[ F_1(\xi) = -\alpha \xi_1 + h(\xi_3); \quad F_2(\xi) = -\beta \xi_2 + \xi_1; \quad F_3(\xi) = -\gamma \xi_3 + \xi_2; \]
one may easily check that (1), (3) and (4) are satisfied whenever \( \Gamma \) is any nonempty subset of \{1, 2, 3\}. From the Theorem we may conclude, for example, if at least one of the nonnegative initial values \( X_1, X_2 \) or \( X_3 \) is nontrivial, the solution \((u(t))_3^1\) to the reaction-diffusion system
\[ \frac{\partial}{\partial t} u_1 = d_1 \Delta u_1 - au_1 + h(u_3), \]
\[ \frac{\partial}{\partial t} u_2 = d_2 \Delta u_2 - \beta u_2 + u_1, \quad (t, x) \in (0, \infty) \times \Omega, \]
\[ \frac{\partial}{\partial t} u_3 = d_3 \Delta u_3 - \gamma u_3 + u_2, \]
\[ (u(t))_3^1 = (X(t))_3^1 \quad \text{for} \ t = 0, \ x \in \Omega, \]
\[ u_i = 0 \quad \text{for} \ t > 0, \ y \in \partial \Omega, \text{and} \ i = 1, 2, 3, \]
satisfies \( u_i(t, x) > 0 \) for all \( t > 0, \ x \in \Omega \) and \( i = 1, 2, 3 \). Here \( d_i > 0 \) for \( i = 1, 2, 3 \) and \( \Delta \) is the Laplacian on \( \Omega \). Observe that the nonlinearity \( F \) is quasimonotone in the case of positive feedback, but not in the case of
negative feedback. Also, since the results of [6] depend not only on the quasimonotonicity of \( F \) but also the irreducibility of the jacobian \( F'(\theta) \), one sees that the results of [6] establish the strict positiveness of solutions to (9) only in the case of positive feedback with \( \sigma = 1 \).

Remark. If one assumes that (PS) has a nonnegative solution \( \tilde{u} \) on \([0, T) \times \Omega\), then the Theorem remains valid under less restrictive assumptions on the smoothness of \( F \) and \( \partial \Omega \). Note that it is necessary only to be assured that each component of \( \tilde{u} \) is either strictly positive or identically zero for the Theorem to hold.

References


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