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A MAXIMUM PRINCIPLE FOR SEMILINEAR PARABOLIC SYSTEMS[1]

ROBERT H. MARTIN, JR.

ABSTRACT. We develop a criterion insuring that every component of the solution to a system of semilinear parabolic equations is strictly positive for positive time. This criterion involves the strict (component-wise) positiveness of solutions to a related ordinary differentiable system.

In this note we present a result concerning the strict positiveness of solutions $\vec{u} = (u_1, \ldots, u_m)$ to the following system of weakly coupled parabolic equations

$$\frac{\partial}{\partial t} u_k(t, x) = L_k u_k(t, x) + F_k(t, x, \vec{u}(t, x), \Delta u_k(t, x)),$$

$$t > 0, x \in \Omega, y \in \partial\Omega, k = 1, \ldots, m,$$

$$B_k u_k(t, y) = 0, \qquad u_k(0, x) = \chi_k(x) \geqslant 0. \tag{PS}$$

Here $\Omega \subset \mathbf{R}^n$ $(n \geqslant 1)$ is a bounded domain with smooth boundary, $\partial\Omega$, and Δ is the gradient operator (with respect to $x \in \mathbf{R}^n$). Also, for each $k \in \{1, \ldots, m\}$, L_k is a uniformly elliptic operator with the representation

$$L_k \sim \sum_{i,j=1}^{n} \frac{\partial}{\partial x_i}\left(a_{k,ij}(x) \frac{\partial}{\partial x_j}\right)$$

with real-valued, smooth coefficient functions $a_{k,ij} = a_{k,ji}$; and B_k is a boundary operator on $(0, \infty) \times \partial\Omega$ of the form

$$B_k u_k(t, y) = b_k(y) u_k(t, y) + \delta_k \frac{\partial}{\partial \nu} u_k(t, y)$$

where ν is the outward normal on $\partial\Omega$ and either $\delta_k = 0$ and $b_k(y) \equiv 1$ on $\partial\Omega$, or $\delta_k = 1$ and $b_k(y) \geqslant 0$ on $\partial\Omega$. Moreover, the real-valued function F_k is C^2 on $[0, \infty) \times \bar{\Omega} \times \mathbf{R}^m \times \mathbf{R}^n$ and for each $R > 0$ there are numbers $M > 0$ and $\gamma \in [0, 2)$ such that

$$|F_k(t, x, \xi, \eta)| \leqslant M(1 + |\eta|^\gamma)$$

whenever $t \in [0, R]$, $x \in \bar{\Omega}$, $\eta \in \mathbf{R}^n$, and $\xi \in \mathbf{R}^m$ with $\|\xi\| \leqslant R$.

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The purpose of this note is to obtain a criterion insuring the strict positiveness of *every* component for solutions to (PS). Our criterion is based on the behavior of solutions to a related system of ordinary differential equations in $\mathbf{R}^m$. Let θ denote the zero vector in $\mathbf{R}^m$ and for each $\xi = (\xi_i)_1^m$ and $\eta = (\eta_i)_1^m$ in $\mathbf{R}^m$ write $\xi \geqslant \eta$ only in case $\xi_i \geqslant \eta_i$ for $i = 1, \ldots, m$, and write $\xi \gg \eta$ only in case $\xi_i > \eta_i$ for $i = 1, \ldots, m$. Also, let θ denote the zero of $\mathbf{R}^n$. It is assumed that $F = (F_k)_1^m$ satisfies the following type of quasipositive condition:

$$\begin{aligned} &\text{if } k \in \{1, \ldots, m\} \text{ and } \xi \in \mathbf{R}^m \text{ with } \xi \geqslant \theta \text{ and } \xi_k = 0, \\ &\text{then } F_k(t, x, \xi, \theta) \geqslant 0 \text{ for all } (t, x) \in [0, \infty) \times \Omega. \end{aligned} \tag{1}$$

Throughout this note it is assumed that x_0 is a (fixed) member of Ω and that $g = (g_k)_1^m$ is defined on $[0, \infty) \times \mathbf{R}^m$ by

$$g_k(t, \xi) = F_k(t, x_0, \xi, \theta) \quad \text{for } (t, \xi) \in [0, \infty) \times \mathbf{R}^m \text{ and } k = 1, \ldots, m. \tag{2}$$

Our comparison system of ordinary differential equations is

$$z'(t) = g(t, z(t)), \qquad t > 0. \tag{ODE}$$

From (1) we see that if $k \in \{1, \ldots, m\}$ and $\xi \geqslant \theta$ with $\xi_k = 0$, then $g_k(t, \xi) \geqslant \theta$, and therefore it is easily deduced that if z is a solution to (ODE) with $z(t_0) \geqslant \theta$, then $z(t) \geqslant \theta$ for all $t \geqslant t_0$. Our second main supposition is

$$\begin{aligned} &\Gamma \text{ is a subset of } \{1, \ldots, m\} \text{ with the property that for each solution} \\ &z = (z_i)_1^m \text{ to (ODE), the conditions } z(t_0) \geqslant \theta \text{ and } z_i(t_0) > 0 \text{ for } i \in \Gamma \\ &\text{imply } z(t) \gg \theta \text{ for all } t > t_0. \end{aligned} \tag{3}$$

One should note that if $z(t_0) \gg \theta$, then $z(t) \gg \theta$ for all $t \geqslant t_0$, and hence (3) is always satisfied with $\Gamma = \{1, \ldots, m\}$. If each solution z has the property that $z(t_0) \geqslant \theta$ implies $z(t) \gg \theta$ for all $t > t_0$, then (3) holds with Γ the empty set.

The final assumption is technical, but is less restrictive than requiring F to be quasimonotone:

$$\begin{aligned} &\text{if } t \geqslant 0, k \in \{1, \ldots, m\} - \Gamma, \text{ and } \xi \geqslant \theta \text{ is such that } \xi_k = 0 \text{ and} \\ &F_k(t, x_0, \xi, \theta) = 0, \text{ then } F_k(t, x_0, \eta, \theta) = 0 \text{ for all } \theta \leqslant \eta \leqslant \xi. \end{aligned} \tag{4}$$

Observe that (4) is a consequence of (1) whenever

$$\partial / \partial \xi_j F_i(t, x_0, \xi, \theta) \geqslant 0 \quad \text{for } i \neq j \text{ (i.e. whenever } F \text{ is quasimonotone).}$$

THEOREM. *Suppose Γ is a subset of $\{1, \ldots, m\}$ and* (1), (3) *and* (4) *are satisfied. Suppose also that the nonnegative initial function $\chi = (\chi_k)_1^m$ is continuous on $\bar{\Omega}$ and that χ_k is nontrivial for each $k \in \Gamma$. Then there is a $T > 0$ such that the solution $\vec{u} = (u_k)_1^m$ to* (PS) *exists on $[0, T) \times \Omega$ and satisfies $\vec{u}(t, x) \gg \theta$ for all $(t, x) \in (0, T) \times \Omega$.*

For our proof we use the following result:

LEMMA. *For each continuous, nonnegative $\chi = (\chi_k)_1^m$ on $\bar{\Omega}$ there is a $T > 0$ such that* (PS) *has a solution $\vec{u} = (u_k)_1^m$ on $[0, T) \times \Omega$ with the property that if*

$k \in \{1, \ldots, m\}$, *then either* $u_k(t, x) \equiv 0$ *on* $(0, T) \times \Omega$ *or* $u_k(t, x) > 0$ *on* $(0, T) \times \Omega$.

We first give the proof of the theorem and then indicate the proof of this lemma.

PROOF OF THEOREM. Let $\vec{u} = (u_k)_1^m$ be the solution to (PS) guaranteed by the Lemma and suppose, for contradiction, that $u_k(t_1, x_1) = 0$ for some $(t_1, x_1) \in (0, T) \times \Omega$ and some k. Using the Lemma again, we see that if $\Gamma_0 = \{k\colon u_k \equiv 0 \text{ on } (0, T) \times \Omega\}$ and $\Gamma_1 = \{k\colon u_k > 0 \text{ on } (0, T) \times \Omega\}$ then $\Gamma_0 \neq \varnothing$, $\Gamma_0 \cup \Gamma_1 = \{1, \ldots, m\}$, $\Gamma_1 \supset \Gamma$, and $\Gamma_0 \cap \Gamma_1 = \varnothing$. It is immediate from (PS) that

$$F_k(t, x, \vec{u}(t, x), \theta) \equiv 0 \quad \text{on } (0, T) \times \Omega \text{ and } k \in \Gamma_0. \tag{5}$$

Choose $0 < a < b < T$ and select $\xi = (\xi_i)_1^m \in \mathbf{R}^m$ such that $\xi_i = 0$ for $i \in \Gamma_0$ and $0 < 2\xi_i < u_i(t, x_0)$ for $i \in \Gamma_1$ and $t \in [a, b]$. From (4) and (5) we have

$$F_k(t, x_0, \eta, \theta) = 0 \quad \text{for } t \in [a, b], \theta \leqslant \eta \leqslant 2\xi, k \in \Gamma_0. \tag{6}$$

Now define the function $z = (z_i)_1^m$ on $[a, b]$ as follows: $z_i(t) \equiv 0$ on $[a, b]$ for $i \in \Gamma_0$ and $\{z_i\colon i \in \Gamma_1\}$ satisfies the initial value problem

$$z_i'(t) = F_i(t, x_0, z(t), \theta), \qquad t \in [a, b), z_i(a) = \xi_i, i \in \Gamma_1. \tag{7}$$

Since $\xi_i > 0$ for $i \in \Gamma_1$, choose $c \in (a, b)$ such that $0 < z_i(t) < 2\xi_i$ for all $t \in [a, c]$ and $i \in \Gamma_1$, and then note that

$$0 = F_i(t, x_0, z(t), \theta), \qquad t \in [a, c), z_i(a) = \xi_i = 0, i \in \Gamma_0, \tag{7$'$}$$

by (6). From the definition of g (see (2)) it is immediate from (7) and (7)′ that z is a solution to (ODE) on $[a, c)$ with $z_i(a) > 0$ for $i \in \Gamma_1 \supset \Gamma$ and $z_i(t) \equiv 0$ for $i \in \Gamma_0$. This contradicts assumption (3) and we conclude that Γ_0 must be empty. This proves the Theorem once the Lemma is established.

PROOF OF LEMMA. The quasipositive assumption (1) along with a weak form of the maximum principle for parabolic equations implies that the solution $\vec{u} = (u_k)_1^m$ satisfies $\vec{u}(t, x) \geqslant \theta$ on $[0, T) \times \Omega$ for some $T > 0$ (see, e.g., the techniques in Amann [**1**], Lemmert [**4**], Lightbourne and Martin [**5**], and Volkmann [**8**]). Fix a number k in $\{1, \ldots, m\}$ and for $(t, x) \in [0, T) \times \Omega$, $r \in \mathbf{R}$, and $l \in \{1, \ldots, n\}$ define

$$w_{k,t,x,r} = (\xi_i)_1^m \quad \text{where } \xi_k = r \text{ and } \xi_i = u_i(t, x) \text{ for } i \neq k$$

and

$$q_{k,t,x,l,r} = (\eta_j)_1^n \quad \text{where } \eta_l = r, \eta_j = 0 \text{ for } j < l \text{ and}$$
$$\eta_j = \partial/\partial x_j u_k(t, x) \quad \text{for } j > l.$$

Now define

$$\alpha_k(t, x) = u_k(t, x)^{-1} \int_0^{u_k(t, x)} \partial/\partial \xi_k F_k(t, x, w_{k,t,x,r}, \Delta u_k(t, x))\, dr$$

and

$$\beta_{k,l}(t, x) = h_l(t, x)^{-1} \int_0^{h_l(t,x)} \partial/\partial q_l F_k(t, x, w_{k,t,x,0}, q_{k,t,x,l,r})\, dr$$

where $h_l(t, x) \equiv \partial/\partial x_l u_k(t, x)$ $(l = 1, \ldots, n)$. Then

$$F_k(t, x, \vec{u}(t, x), \Delta u_k(t, x)) = F_k(t, x, w_{k,t,x,0}, \theta) + \alpha_k(t, x) u_k(t, x) + \sum_{l=1}^{m} \beta_{k,l}(t, x) \partial/\partial x_l u_k(t, x)$$

and since $F_k(t, x, w_{k,t,x,0}, \theta) \geqslant 0$ by (1) it follows from (PS) that

$$\partial/\partial t u_k(t, x) \geqslant L_k u_k(t, x) + \alpha_k(t, x) u_k(t, x) + \sum_{l=1}^{n} \beta_{k,l}(t, x) \partial/\partial x_l u_k(t, x)$$

for all $(t, x) \in (0, T) \times \Omega$. Since $u_k \geqslant 0$ on $(0, T) \times \Omega$ we have from a strong form of the maximum principle [7, pp. 173 and 175] that T can be chosen so that either $u_k \equiv 0$ on $(0, T) \times \Omega$ or $u_k > 0$ on $(0, T) \times \Omega$. This completes the proof indication of the Lemma.

As an illustration of this result, we consider the mathematical model of a cellular control process with either positive or negative feedback. (See Griffith [2], [3].) This model is the system of three ordinary differential equations

$$\begin{aligned} z_1' &= -\alpha z_1 + h(z_3), & z_1(0) &\geqslant 0, \\ z_2' &= -\beta z_2 + z_1, & z_2(0) &\geqslant 0, \\ z_3' &= -\gamma z_3 + z_2, & z_3(0) &\geqslant 0, \end{aligned} \tag{8}$$

where α, β, γ are positive constants and the function h is defined on $[0, \infty)$ by either $h(r) = r^\sigma(1 + r^\sigma)^{-1}$ (positive feedback) or $h(r) = (1 + r^\sigma)^{-1}$ (negative feedback), and $\sigma \geqslant 1$ is a constant. Defining $F = (F_i)_1^3$ by the right side of (8):

$$F_1(\xi) = -\alpha\xi_1 + h(\xi_3); \quad F_2(\xi) = -\beta z_2 + z_1; \quad F_3(\xi) = -\gamma z_3 + z_2;$$

one may easily check that (1), (3) and (4) are satisfied whenever Γ is any nonempty subset of $\{1, 2, 3\}$. From the Theorem we may conclude, for example, if at least one of the nonnegative initial values χ_1, χ_2 or χ_3 is nontrivial, the solution $(u_i)_1^3$ to the reaction-diffusion system

$$\begin{aligned} \partial/\partial t u_1 &= d_1 \Delta u_1 - \alpha u_1 + h(u_3), \\ \partial/\partial t u_2 &= d_2 \Delta u_2 - \beta u_2 + u_1, \qquad (t, x) \in (0, \infty) \times \Omega, \\ \partial/\partial t u_3 &= d_3 \Delta u_3 - \gamma u_3 + u_2, \\ (u_i)_1^3 &= (\chi_i)_1^3 \quad \text{for } t = 0, x \in \Omega, \\ u_i &= 0 \quad \text{for } t > 0, y \in \partial\Omega, \text{ and } i = 1, 2, 3, \end{aligned}$$

satisfies $u_i(t, x) > 0$ for all $t > 0$, $x \in \Omega$ and $i = 1, 2, 3$. Here $d_i > 0$ for $i = 1, 2, 3$ and Δ is the Laplacian on Ω. Observe that the nonlinearity F is quasimonotone in the case of positive feedback, but not in the case of

negative feedback. Also, since the results of [**6**] depend not only on the quasimonotonicity of F but also the irreducibility of the jacobian $F'(\theta)$, one sees that the results of [**6**] establish the strict positiveness of solutions to (9) only in the case of positive feedback with $\sigma = 1$.

REMARK. If one *assumes* that (PS) has a nonnegative solution $\vec{u}$ on $[0, T) \times \Omega$, then the Theorem remains valid under less restrictive assumptions on the smoothness of F and $\partial\Omega$. Note that it is necessary only to be assured that each component of $\vec{u}$ is either strictly positive or identically zero for the Theorem to hold.

REFERENCES

1. H. Amann, *Invariant sets and existence theorems for semi-linear parabolic and elliptic systems* (to appear).

2. J. S. Griffith, *Mathematics of cellular control processes*. I. *Negative feedback to one gene*, J. Theoret. Biol. **20** (1968), 202–208.

3. _____, *Mathematics of cellular control processes*. II. *Positive feedback to one gene*, J. Theoret. Biol. **20** (1968), 209–216.

4. R. Lemmert, *Über die Invarianz einer konvexen Menge in bezug auf Systeme von gewöhnlichen, parabolischen und elliptischen Differentialgleichungen*, Math. Ann. **230** (1977), 49–56.

5. J. H. Lightbourne and R. H. Martin, *Relatively continuous nonlinear perturbations of analytic semigroups*, JNA-TMA **1** (1977), 277–292.

6. R. H. Martin, Jr., *Asymptotic stability and critical points for nonlinear quasimonotone parabolic systems*, J. Differential Equations (to appear).

7. M. H. Protter and H. F. Weinberger, *Maximum principles in differential equations*, Prentice-Hall, Englewood Cliffs, N. J., 1967.

8. P. Volkmann, *Über die positive Invarianz einer abgeschlossenen Teilmenge eines Banachschen Raumes bezüzlich der Differentialgleichung* $u' = f(t, u)$, J. Reine Angew. Math. **285** (1976), 59–65.

DEPARTMENT OF MATHEMATICS, NORTH CAROLINA STATE UNIVERSITY, RALEIGH, NORTH CAROLINA 27607