ON Ext$^*_E(K_*(K), K_*(MU))$

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Abstract. In this note a spectral sequence is constructed which is used to prove the existence of elements of nonzero homological degree in $\text{Ext}^*_E(K_*(K), K_*(MU))$. Examples of specific elements in terms of representatives in the cobar construction are given.

1. Introduction and statement of results. If $E$ is a multiplicative spectrum such that $E_*(E)$ is a flat $E_*(E)$-module, then, under certain conditions listed in [2, pp. 317–318], there is, for spectra $X$ and $Y$, a spectral sequence

$$\text{Ext}^*_E(E_*(X), E_*(Y)) \Rightarrow [X, Y]^E_*,$$

where $[X, Y]^E_*$ denotes the set of morphisms in a certain category of spectra. Roughly speaking, $[X, Y]^E$ is $[X, Y]$ modulo those maps which $E$-theory cannot detect. The Ext groups are computed in the category of comodules over $E_*(E)$.

Taking $E = K$, $X = S^0$, $Y = MU$ we have a spectral sequence whose $E_2$-term is $\text{Ext}^*_E(K_*(K), K_*(MU))$. However, $K$ is not connective, and so assumption (iii)(b) on p. 317 of [2] does not hold; thus we do not have the convergence part of Theorem 15.1 of Part III of [2]. On the other hand the edge-homomorphism

$$\pi_*(MU) \to \text{Ext}^*_E(K_*(K), K_*(MU)) = PK_*(MU)$$

is an isomorphism; this is equivalent to the Hattori-Stong theorem; see [2, Theorem 14.1 of Part II] or [7, Theorem 20.34].

It seemed natural to suspect that

$$\text{Ext}^*_E(K_*(K), K_*(MU)) = 0 \quad \text{for } p \neq 0$$

and that the spectral sequence collapsed. The hope was that, if one could prove this, the convergence problem might be circumvented and new proofs of the Hattori-Stong theorem and Milnor's theorem on the structure of $\pi_*(MU)$ obtained. See [7, pp. 516–517].

The purpose of this note is to prove that this conjecture is false. That is, we show that $\text{Ext}^*_E(K_*(K), K_*(MU))$ contains nonzero elements of positive homological degree. Our proof involves considering $K_*(MU)$ at the same time as a left $K_*(K)$-comodule and as a right $MU_*(MU)$-comodule.
THEOREM 1. There is a (trigraded) spectral sequence with $E_2$-term

$$\text{Ext}_{K_*}^{p,q}(\pi_*(K), K_*(MU))$$

which converges to $\text{Ext}_{K_*}^{p,q}(\pi_*(K), K_*(MU))$.

Here the $E_2$-term is calculated as follows. The inner Ext groups are computed in the category of left $K_*(K)$-comodules; each $\text{Ext}_{K_*}^{p,q}(\pi_*(K), K_*(MU))$ is a right $MU_*(MU)$-comodule and the outer Ext groups are defined in the category of such comodules. More details of this algebra will be found below.

COROLLARY. $\text{Ext}_{K_*}^{p,q}(\pi_*(K), K_*(MU))$ is nonzero for some positive $q$.

PROOF OF COROLLARY FROM THEOREM 1. If the Corollary is false the spectral sequence of the Theorem has

$$E_2^{p,q} = \begin{cases} \text{Ext}_{MU_*}^{p,q}(\pi_*(MU), \pi_*(MU)), & q = 0, \\ 0, & q \neq 0. \end{cases}$$

(Note that this Ext group is taken in the category of right $MU_*(MU)$-comodules; but it is clearly isomorphic to the usual $E_2$-term of the Adams-Novikov spectral sequence.)

Now the spectral sequence collapses giving

$$\text{Ext}_{MU_*}^{p,q}(\pi_*(MU), \pi_*(MU)) = \text{Ext}_{K_*}^{p,q}(\pi_*(K), \pi_*(K)).$$

But this contradicts various calculations made of these groups. For example,

$$\text{Ext}_{MU_*}^{1,k}(\pi_*(MU), \pi_*(MU)) = 0 \quad \text{for } k < 0,$$

while

$$\text{Ext}_{K_*}^{1,2n}(\pi_*(K), \pi_*(K)) = Z/m(n)Z \quad \text{for all } n \in Z \quad [5].$$

Alternatively,

$$\text{Ext}_{MU_*}^{1,2n}(\pi_*(MU), \pi_*(MU)) = Z/12Z \quad [6], [8].$$

while

$$\text{Ext}_{K_*}^{1,2n}(\pi_*(K), \pi_*(K)) = Z/24Z.$$
Here $\psi = \psi_{MU}$ is the usual map of [1, Lecture 3], and $\psi'$ is defined by the same construction, but with everything switched from left to right throughout. Thus $\psi'$ is the composition

$$K_*(MU) \xrightarrow{t} MU_*(K) \xrightarrow{\psi_K} MU_*(MU) \otimes_{\pi_*(MU)} MU_*(MU)$$

where the last isomorphism sends $\alpha \otimes \beta$ to $t^*\beta \otimes ca$, $c$ being the canonical antiautomorphism of $MU_*(MU)$, and $t$ the switch map.

The following diagram commutes:

$$
\begin{array}{ccc}
K_*(MU) & \xrightarrow{\psi} & K_*(K) \otimes_{\pi_*(K)} K_*(MU) \\
\downarrow \psi & & \downarrow 1 \otimes \psi' \\
K_*(MU) \otimes_{\pi_*(MU)} MU_*(MU) & \xrightarrow{\psi \otimes 1} & K_*(K) \otimes_{\pi_*(K)} K_*(MU) \otimes_{\pi_*(MU)} MU_*(MU) \\
\end{array}
$$

That is, $\psi$ is a morphism of right $MU_*(MU)$-comodules and $\psi'$ is a morphism of left $K_*(K)$-comodules.

**Lemma.**

$$\text{Ext}_{MU_*(MU)}^p(\pi_*(MU), MU_*(K)) = \begin{cases} 
\pi_*(K), & p = 0, \\
0, & p \neq 0.
\end{cases}$$

**Proof.** By the Conner-Floyd theorem, $MU_*(K) = MU_*(MU) \otimes_{\pi_*(MU)} \pi_*(K)$; thus $MU_*(K)$ is an extended $MU_*(MU)$-comodule and the lemma follows [2, p. 321] or [7, Chapter 19].

It follows that, in terms of right $MU_*(MU)$-comodules,

$$\text{Ext}_{MU_*(MU)}^p(\pi_*(MU), K_*(MU)) = \begin{cases} 
\pi_*(K), & p = 0, \\
0, & p \neq 0.
\end{cases}$$

If $M$ is a left comodule over a two-sided coalgebra $C$ with ground ring $R$, then $\text{Ext}_C(R, M)$ may be computed as the homology of the cobar complex

$$M \xrightarrow{d_0} \bar{C} \otimes_R M \xrightarrow{d_1} \bar{C} \otimes_R \bar{C} \otimes_R M \xrightarrow{d_2} \cdots,$$

where $\bar{C} = \ker\{e: C \to R\}$ and

$$d_i(c_1 \otimes \cdots \otimes c_i \otimes m) = 1 \otimes c_1 \otimes \cdots \otimes c_i \otimes m + (-1)^{i+1} c_1 \otimes \cdots \otimes c_i \otimes m.$$

Similarly, we can form a cobar resolution for right comodules. Let us write $M = K_*(MU)$, $C = K_*(K)$, $R = \pi_*(K)$, $D = MU_*(MU)$ and $S = \pi_*(MU)$. 

ON \( \text{Ext}_{K_s(K)}(\pi_s(K), K_s(MU)) \)

Since the structure maps \( M \overset{\psi}{\to} C \otimes_R M \) and \( M \overset{\psi}{\to} M \otimes_S D \) commute, we may put both cobar complexes together to form a double complex.

\[
\begin{array}{cccc}
\begin{array}{c}
\bar{C} \otimes_R \bar{C} \otimes_R M \\
\uparrow \\
\bar{C} \otimes_R M \\
\uparrow \\
M
\end{array} & \to &
\begin{array}{c}
\bar{C} \otimes_R \bar{C} \otimes_R M \otimes_S D \\
\uparrow \\
\bar{C} \otimes_R M \otimes_S D \\
\uparrow \\
M \otimes_S D
\end{array} & \to & \cdots
\end{array}
\]

Now a double complex \( A \) has associated with it two spectral sequences, whose \( E \)-terms are \( H_1 H_{II} A \) and \( H_{II} H_1 A \), respectively (where \( H_1 \) represents homology with respect to the horizontal differentials and \( H_{II} \) with respect to the vertical differentials), and both of which converge to the total homology \( H(A) \) \([4, \text{pp. 331-332}]\). We will see that the second spectral sequence collapses, allowing us to compute \( H(A) \), and then the first spectral sequence will be that of Theorem 1.

\( C \) and \( D \), and hence \( \bar{C} \) and \( \bar{D} \), are flat modules over \( R \) and \( S \). By the Lemma, the first row is exact, and so all the rows are exact and, thus,

\[
H_1(A)^{p,q} = \begin{cases} 
\bar{C} \otimes_R \cdots \otimes_R \bar{C} & (q \text{ factors}), \\
0 & p = 0, \\
& p \neq 0,
\end{cases}
\]

giving

\[
H_{II} H_1(A)^{p,q} = \begin{cases} 
\text{Ext}_C^q(R, R), & p = 0, \\
0, & p \neq 0.
\end{cases}
\]

We must therefore have \( H(A) = \text{Ext}_C^*(R, R) \).

Turning to the first spectral sequence, we know that the first column has \( \text{Ext}_C^*(R, M) \) as its homology. Hence by the flatness of \( \bar{D} \), the \( q \)th column has homology \( \text{Ext}_C^*(R, M) \otimes_S \bar{D} \otimes_S \cdots \otimes_S \bar{D} \) (\( q \) factors). It follows that the \( E_2 \)-term of the first spectral sequence is \( \text{Ext}_D^p(S, \text{Ext}_C^*(R, M)) \). Restoring the notation gives Theorem 1.

3. Nonzero elements in \( \text{Ext}_C^*(K_s(K), K_s(MU)) \). We show here that, in the standard notation of \([2]\) and \([7]\),

\[
(u^n - v^n)/m(n) \otimes 1 \in K_s(K) \otimes_{\pi_s(K)} K_s(MU)
\]

is a cycle in the cobar complex for computing \( \text{Ext}_{K_s(K)}(\pi_s(K), K_s(MU)) \), which, for \( n \) negative, gives an element of order \( m(n) \) in \( \text{Ext}^{1,2n}_1 \), and, for \( n = 2 \), gives an element of order 2 in \( \text{Ext}^{1,4}_1 \).

We can see this as follows. \((u^n - v^n)/m(n) \otimes 1\) is a cycle because \((u^n - v^n)/m(n) \in K_{2n}(K)\) is primitive \([5]\).

\[
d_0: K_s(MU) \to K_s(K) \otimes_{\pi_s(K)} K_s(MU)
\]
is defined by $d_0\alpha = \psi\alpha - 1 \otimes \alpha$. Write

$$K_*(MU) = \pi_*(K)[b_1, b_2, \ldots] = \mathbb{Z}[t, t^{-1}][b_1, b_2, \ldots];$$

then $d_0t^n = (u^n - v^n) \otimes 1$.

Now suppose that $k[(u^n - v^n)/m(n) \otimes 1] = 0$ in $\text{Ext}^{1,2n}$, so that there exists $\alpha \in K_*(MU)$ such that

$$d_0\alpha = k(u^n - v^n)/m(n) \otimes 1.$$

Then $d_0(m(n)\alpha - kt^n) = 0$, and $m(n)\alpha - kt^n$ is an element of $PK_{2n}(MU)$, nonzero unless $\alpha = kt^n/m(n)$, and $k$ is a multiple of $m(n)$. But if $n$ is negative, $PK_{2n}(MU) \cong \pi_{2n}(MU) = 0$.

Similarly, $[(u^2 - v^2)/24 \otimes 1]$ has order 2 in $\text{Ext}^{1,4}$. For Proposition 17.38 of [7] gives

$$d_0(2b_2 - b_1 + tb_1) = (v^2 - u^2)/12 \otimes 1.$$

If there exists $\alpha \in K_*(MU)$ such that $d_0\alpha = (u^2 - v^2)/24 \otimes 1$, then, as before, $2\alpha + 2b_2 - b_1^2 + tb_1 \in PK_*(MU)$. But (see [7, p. 437]) $PK_*(MU)$ is generated by $3b_2 - 2b_1^2 + tb_1$ and $(2b_1 + t)^2$. Then we have, for some integers $A$ and $B$,

$$2\alpha + 2b_2 - b_1^2 + tb_1 = A(3b_2 - 2b_1^2 + tb_1) + B(2b_1 + t)^2,$$

and the coefficient of $b_1^2$ in $2\alpha$ is $4B - 2A + 1$, giving a contradiction. Hence $[(u^2 - v^2)/24 \otimes 1]$ has order 2.

References

2. ———, *Stable homotopy and generalized homology*, Univ. of Chicago Math. Lecture Notes, 1974.

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