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REMARKS ON THE REPRESENTATION OF ZERO BY SOLUTIONS OF DIFFERENTIAL EQUATIONS

JET WIMP AND DAVID COLTON[1]

ABSTRACT. Numerical evidence from certain problems arising in optics seems to indicate Fourier-Bessel series which converge to zero in $(1-\delta, 1]$ also converge to zero in $[1, 1+\delta)$, an interval which lies outside the range of orthogonality of the Bessel functions. Here we demonstrate this as a corollary of a result on series of functions which satisfy a general Sturm-Liouville equation.

Sometimes it is necessary to determine the behaviour of Fourier-Bessel series

$$\sum b_n J_\nu(\rho_n x), \quad \nu > -\frac{1}{2}, \quad J_\nu(\rho_n) = 0$$

outside the interval $[0, 1]$, the interval of orthogonality of the functions $J_\nu(\rho_n x)$. For instance in a certain problem concerning circularly symmetric positive filters (see [2]) it was given that the above series converged to zero for $x \in [1-\delta, 1]$. In order to guarantee a sufficient spacing in the ring pattern, it was necessary to show that it converged to zero for $x \in [1, 1+\delta]$. Numerical evidence seemed to support this conclusion, and a rigorous proof of this fact has been communicated to the authors by J. Boersma, who based his analysis on the properties of Fourier-Bessel series. In this note we shall show that this result has nothing to do with any special properties (orthogonality, etc.) of Bessel functions, and is in fact true for any sequence of functions satisfying a second order differential equation of Sturm-Liouville type. This result in turn will lead to a rather surprising result on nontrivial representations of zero by infinite series of solutions to ordinary differential equations. In what follows let $\{c_n\}$ be any sequence of real, nonzero constants and $\{y_n\}$ a sequence of functions satisfying

$$Ly_n = c_n^2 y_n, \quad y_n(0) = 0, \quad y_n'(0) = c_n$$

where $L = -d^2/dx^2 + q(x)$ and $q \in C[-\delta, \delta]$. Note that in the analysis which follows the initial condition

$$y_n'(0) = c_n$$

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merely plays the role of a normalization constant.

THEOREM. *If* $\sum |a_n| < \infty$ *and*

$$0 = \sum a_n y_n(x), \quad x \in [0, \delta] \tag{1}$$

then

$$0 = \sum a_n y_n(x), \quad x \in [-\delta, 0]. \tag{2}$$

PROOF. We use the representation [**1**, Appendix IV],

$$y_n(x) = \sin(c_n x) + \int_0^x K(x, t) \sin(c_n t)\, dt, \quad x \in [-\delta, \delta] \tag{3}$$

where $K(x, t)$ is a continuously differentiable function of x and t that is uniquely determined by the function q and the condition $K(0, 0) = 0$. Since K is continuous and $|a_n y_n(x)| \leqslant M|a_n|$,

$$M = 1 + \sup_x \int_0^x |K(x, t)|\, dt, \quad x \in [-\delta, \delta],$$

the series (1) and (2) converge absolutely and uniformly for $x \in [-\delta, \delta]$. Multiplying (3) by a_n and summing shows that

$$0 = u(x) + \int_0^x K(x, t) u(t)\, dt, \quad x \in [0, \delta], u(x) = \sum a_n \sin(c_n x)$$

and by the uniqueness of solutions to Volterra integral equations this means

$$u(x) = 0, \quad x \in [0, \delta].$$

Furthermore

$$y_n(-x) = -\sin(c_n x) + \int_0^x K(-x, -t) \sin(c_n t)\, dt, \quad x \in [0, \delta],$$

so

$$\sum a_n y_n(-x) = -u(x) + \int_0^x K(-x, -t) u(t)\, dt = 0, \quad x \in [0, \delta].$$

This concludes the proof of the Theorem.

Returning now to our Fourier-Bessel series we note that $y_n(x) = x^{1/2} J_\nu(\rho_n x)$ satisfies the equation

$$L y_n = \rho_n^2 y_n$$

with $q(x) = x^{-2}(\nu^2 - \frac{1}{4})$. Hence replacing x by $1 - x$, c_n by ρ_n, and using well-known asymptotic properties of Bessel functions we have

COROLLARY 1. *Let* $\sum n^{-1/2} |b_n| < \infty$,

$$\sum b_n J_\nu(\rho_n x) = 0, \quad x \in [1 - \delta, 1].$$

Then the same holds for $x \in [1, 1 + \delta]$.

From the proof of the Theorem we arrive at the surprising

COROLLARY 2. *Let* $\sum|a_n| < \infty$. *Then*

$$0 = \sum a_n y_n(x), \quad x \in [0, \delta] \tag{4}$$

if and only if

$$0 = \sum a_n \sin(c_n x), \quad x \in [0, \delta]. \tag{5}$$

In other words a nontrivial representation of zero on an interval in terms of one set of functions leads to another nontrivial representation of zero with the same coefficients but different functions in the expansion. One's first reaction is to suspect that all the coefficients a_n *must* be zero if $\sum|a_n| < \infty$ and (4) or (5) is true. But it is quickly seen that this need not be the case if one considers a function $f \in C^2[0, \pi], f \neq 0$, such that $f(\pi) = 0, f(x) = 0$ for $x \in [0, \delta]$ and expands f in a Fourier sine series. In this case we have that $a_n \neq 0$ for all n, $|a_n| = O(n^{-2})$, and (5) is valid for $c_n = n$. It can also be shown that if $f \in C^k[0, 1], f^{(j)}(1) = 0, 0 \leqslant j \leqslant k - 1$, then the Fourier-Bessel coefficients of f are $O(n^{1-k})$. Thus if $k \geqslant 2$ and we make $f(x) > 0$ on $[0, 1 - \delta]$ and zero on $[1 - \delta, 1]$ then $\sum n^{-1/2}|b_n| < \infty$ (see Corollary 1) but clearly b_n will not be zero for all n.

It would be interesting to determine necessary conditions on the sequence $\{c_n\}$ such that $\sum|a_n| < \infty$ and (5), or equivalently (4), imply that all the a_n are zero.

REFERENCES

1. B. Ja. Levin, *Distribution of zeros of entire functions*, Transl. Math. Monographs, vol. 5, Amer. Math. Soc., Providence, R. I., 1964, Theorem 1, p. 427.

2. P. Wiersma, *The realization of circularly symmetric positive filter functions by transparent rings*, Technical Report, Department of Electrical Engineering, Eindhoven University of Technology, Eindhoven, The Netherlands, 1978.

DEPARTMENT OF MATHEMATICS, DREXEL UNIVERSITY, PHILADELPHIA, PENNSYLVANIA 19104

DEPARTMENT OF MATHEMATICS, UNIVERSITY OF DELAWARE, NEWARK, DELAWARE 19711