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PROPERTIES OF REPRESENTATIONS OF BANACH ALGEBRAS WITH APPROXIMATE IDENTITIES

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ABSTRACT. Let A be a Banach algebra with a bounded left approximate identity. The Cohen Factorization Theorem is used to prove several results concerning the representation theory of A.

1. Introduction. Let G be a locally compact group. Certain important types of groups have the property that there exists a dense subspace of $L^1(G)$ such that any irreducible *-representation of $L^1(G)$ on a Hilbert space H maps this subspace into the ideal of Hilbert-Schmidt operators on H [**10**, Theorem 4.5.7.4]. It is natural to ask if it is possible to map all of $L^1(G)$ into a proper ideal of the algebra of completely continuous operators on H. This question was partially answered by L. Baggett in [**1**]. He proved that for an arbitrary locally compact group G, $L^1(G)$ has no infinite dimensional essential *-representation with image contained in the ideal of Hilbert-Schmidt operators [**1**, Theorem, p. 503]. An immediate consequence of this theorem is the result that there must exist operators with infinite dimensional range in the image of any infinite dimensional essential *-representation of $L^1(G)$ [**1**, Corollary 1]. In §3 of this note the question above is completely answered: if A is a Banach algebra with a bounded left approximate identity, and φ is a continuous essential representation of A on a Hilbert space H, then given any ideal I which is proper in the algebra of completely continuous operators on H, there exists $f \in A$ such that $\varphi(f) \notin I$ [Theorem 2]. Furthermore it is shown that the image of a continuous essential representation of A on an infinite dimensional Banach space must contain an operator with infinite dimensional range. These results are derived from the Cohen Factorization Theorem.

Now assume that G is a locally compact abelian group with character group Γ. Let ρ be a nonnegative function in $C_0(\Gamma)$, the continuous complex-valued functions vanishing at ∞ on Γ. P. C Curtis and A. Figà-Talamanca prove that there exists $f \in L^1(G)$ such that $\hat{f}(\gamma) \geqslant \rho(\gamma)$ for all $\gamma \in \Gamma$; see [**5**, (32.47)(b)]. Also, there is a corresponding theorem concerning the rate of decrease of the Fourier transform for compact groups [**5**, (32.47)(a)]. We prove two theorems dealing with the slow rate of decrease of certain maps on

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a Banach algebra that has a bounded left (or right) approximate identity. Again, the proofs are applications of the Cohen Factorization Theorem.

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2. Preliminaries. All of the results in this note concern Banach algebras with bounded left or right approximate identities [**5**, Definition (28.51)]. Any C^*-algebra has a bounded approximate identity, and the same holds for $L^1(G)$ or any Beurling subalgebra of $L^1(G)$ [**9**], G a locally compact group. We use the following notation. Let X be a linear space. If S is a collection of operators on X and W is a subset of X, then let $SW = \{T(w): T \in S, w \in W\}$. Similarly, if C and D are two subsets of an algebra B, then let $CD = \{fg: f \in C, g \in D\}$. When W is a subset of a normed linear space X, then LS(W) and CLS(W) denote the linear span, and the closed linear span, of W in X, respectively. In the next section we use the following simple consequence of the Cohen Factorization Theorem.

COROLLARY (OF THE COHEN FACTORIZATION THEOREM). *Let A be a Banach algebra with a bounded left approximate identity. Let B be a Banach algebra, and assume that R is a subset of B with $RB \subset R$. If φ is a continuous homomorphism of A into B, then either* $\mathrm{CLS}(\varphi(A)B) \subset R$ *or there exists $f \in A$ such that $\varphi(f) \notin R$.*

PROOF. Let A act on B by the rule

$$f \cdot g = \varphi(f)g \qquad (f \in A, g \in B).$$

Then

$$\|f \cdot g\|_B \leqslant \|\varphi\| \, \|f\|_A \, \|g\|_B \qquad (f \in A, g \in B).$$

Thus, B is a left Banach A-module [**5**, Definition (32.14)]. By the Cohen Factorization Theorem [**5**, Theorem (32.22)]

$$A \cdot B = \varphi(A)B = \mathrm{CLS}(\varphi(A)B).$$

Then either

$$\varphi(A) \not\subset R \quad \text{or} \quad \mathrm{CLS}(\varphi(A)B) = \varphi(A)B \subset R.$$

Clearly a similar statement can be made when A has bounded right approximate identity and $BR \subset R$.

The author would like to thank Professor H. G. Feichtinger of Universität Wien who kindly communicated to us a version of this corollary.

3. Representation theory. Let X be a Banach space. We denote the algebra of all bounded linear operators on X by $B(X)$, and the ideal of all operators in $B(X)$ with finite dimensional range by $F(X)$. Let X^* be the dual space of X. For $x \in X$ and $\alpha \in X^*$, let $x \otimes \alpha$ be the operator in $F(X)$ defined by

$$(x \otimes \alpha)(y) = \alpha(y)x \qquad (y \in X).$$

Note that $\|x \otimes \alpha\| = \|x\| \, \|\alpha\|$ and

$$F(X) = \mathrm{LS}\{x \otimes \alpha : x \in X, \alpha \in X^*\}.$$

A representation φ of an algebra A on X is an algebra homomorphism of A into $B(X)$. We use the notation (φ, X) to indicate the representation space X and the homomorphism φ. The representation (φ, X) is essential if $X = \mathrm{CLS}(\varphi(A)X)$.

LEMMA 1. *Let (φ, X) be an essential representation of an algebra A. Let $B = B(X)$. Then $F(X) \subset \mathrm{CLS}(\varphi(A)B)$.*

PROOF. Let $x \in X$, $\alpha \in X^*$. Let $\varepsilon > 0$ be arbitrary. Choose $\{x_1, \ldots, x_n\} \subset X$ and $\{f_1, \ldots, f_n\} \subset A$ such that

$$\left\| \sum_{k=1}^{n} \varphi(f_k) x_k - x \right\| < \varepsilon.$$

Then

$$\left\| \sum_{k=1}^{n} \varphi(f_k)(x_k \otimes \alpha) - x \otimes \alpha \right\| = \left\| \sum_{k=1}^{n} \varphi(f_k) x_k - x \right\| \|\alpha\| < \varepsilon \|\alpha\|.$$

This proves the lemma.

THEOREM 2. *Let A be a Banach algebra with bounded left approximate identity. Let (φ, X) be a continuous essential representation of A. Assume that R is a subset of $B = B(X)$ such that $BR \subset R$ and the closure of $F(X)$ is not in R. Then there exists $f \in A$ such that $\varphi(f) \notin R$.*

PROOF. By the corollary (in §2) either there exists $f \in A$ such that $\varphi(f) \notin R$ or $\mathrm{CLS}(\varphi(A)B) \subset R$. But the latter alternative is impossible since by Lemma 1,

$$F(X) \subset \mathrm{CLS}(\varphi(A)B),$$

and by hypothesis the closure of $F(X)$ is not in R.

The next corollary generalizes a result of L. Baggett [**1**, Corollary 1].

COROLLARY 3. *Let A be a Banach algebra with a bounded left or right approximate identity. Assume that (φ, X) is a continuous essential representation of A such that $\varphi(A) \subset F(X)$. Then X is finite dimensional.*

PROOF. Let $R = F(X)$. If X were infinite dimensional, then $F(X)$ would be a proper subspace of its closure. But then by Theorem 2,

$$\varphi(A) \not\subset R = F(X),$$

which contradicts our hypothesis.

Next we apply Theorem 2 to the case where the representation space is an infinite dimensional Hilbert space H, and R is a union of certain norm ideals in $B(H)$. For each p, $1 \leqslant p \leqslant \infty$, let C_p denote the usual norm ideal of completely continuous operators on H [**3**, Definition 1, p. 1089]. Then

$R = \bigcup \{C_p: 1 \leqslant p < \infty\}$ is a proper ideal in C_∞, the algebra of all completely continuous operators on H. The following result is an immediate consequence of Theorem 2.

COROLLARY 4. *Assume the notation above. Let A be a Banach algebra with a bounded left or right approximate identity. Let (φ, H) be a continuous essential representation of A. Then there exists $f \in A$ such that $\varphi(f) \notin C_p$, $1 \leqslant p < \infty$.*

Corollary 4 is a generalization of [**1**, Theorem, p. 503].

REMARK. Let A_0 be a closed subalgebra of A containing a bounded left approximate identity of A. Assume that (φ, X) is an essential representation of A. Then φ restricted to A_0 is essential on X since

$$\begin{aligned}\mathrm{CLS}(\varphi(A_0)X) &\supset \mathrm{CLS}(\varphi(A_0)\varphi(A)X)\\ &= \mathrm{CLS}(\varphi(A_0A)X) = \mathrm{CLS}(\varphi(A)X).\end{aligned}$$

Thus conclusions concerning A can be replaced by conclusions concerning A_0 in many cases.

To take a specific example where this remark applies, assume that G is a locally compact SIN-group [7]. Then there exists a bounded approximate identity for $L^1(G)$ which is contained in $Z(L^1(G))$, the center of $L^1(G)$, [7, Proposition, p. 614]. Then we have the following version of Corollary 4.

COROLLARY 5. *Assume that G is a* SIN-*group. Let (φ, H) be a continuous essential in finite dimensional representation of $L^1(G)$. Then there exists $f \in Z(L^1(G))$ such that $\varphi(f) \notin C_p$ for $1 \leqslant p < \infty$.*

4. Applications to properties of slow decrease. As indicated in the introduction, the Fourier transform on L^1 of a locally compact abelian or compact group has an arbitrarily slow rate of decrease to zero at infinity. In this section we give two generalizations of this phenomenon.

THEOREM 6. *Let A be a Banach algebra with a left (or right) bounded approximate identity. Let (π_n, X_n) be a uniformly bounded sequence of nontrivial representations of A. Assume that $\{\rho_n\}$ is a sequence of positive numbers such that* $\lim_{n\to\infty} \rho_n = 0$.

Then there exists $f \in A$ such that $\|\pi_n(f)\| \geqslant \rho_n$ for $n \geqslant 1$.

PROOF. Let B be the set of all sequences $\{T_n\}$ where $T_n \in B(X_n)$ for $n \geqslant 1$ and $\{\|T_n\|\}$ is a bounded sequence. B is a Banach algebra with coordinatewise operations and norm

$$\|\{T_n\}\| = \sup\{\|T_n\|: n \geqslant 1\}.$$

Define a map $\varphi: A \to B$ by $\varphi(f) = \{\pi_n(f)\}$. Then φ is a continuous algebra homomorphism of A into B.

It is easy to see that $\mathrm{CLS}(\varphi(A)B)$ contains every sequence $\{S_n\}$ such that $S_n \in \pi_n(A)$ for all n and $\|S_n\| \to 0$ as $n \to \infty$. Choose $\{S_n\} \subset \mathrm{CLS}(\varphi(A)B)$

such that $\|S_n\| = \rho_n$, $n \geqslant 1$. Since

$$\mathrm{CLS}(\varphi(A)B) = \varphi(A)B$$

by the Cohen Factorization Theorem, there exists $g \in A$ and $T = \{T_n\} \subset B$ such that

$$\pi_n(g)T_n = S_n \qquad (n \geqslant 1).$$

Set $f = \|T\| g$. Then

$$\|\pi_n(f)\| = \|T\| \, \|\pi_n(g)\| \geqslant \|\pi_n(g)T_n\| = \|S_n\| = \rho_n.$$

Let B be a C^*-algebra with Hausdorff structure space Γ (Γ is the set of all primitive ideals of B with the hull-kernel topology [2, p. 59]). In this case B can be represented as an algebra of functions on Γ which we now describe. For $f \in B$ and $P \in \Gamma$ we use the notation

$$\hat{f}(P) = f + P \in B/P.$$

The norm

$$\|f\|_\infty = \sup\{\|\hat{f}(P)\|: P \in \Gamma\}$$

is a C^*-norm on B, and therefore $\|f\|_\infty = \|f\|_B$ for $f \in B$ [2, 1.37 and 1.81]. Let $C_0(\Gamma)$ be the algebra of continuous real-valued functions vanishing at ∞ on Γ. By [6, Theorem 4.1 and Remark, p. 235] Γ is locally compact and the function $P \to \|\hat{f}(P)\|$ is in $C_0(\Gamma)$. Furthermore, by the Dauns-Hofmann Theorem [4], if $g \in C_0(\Gamma)$ and $f \in B$, then $h = gf$ is in B in the sense that

$$\hat{h}(P) = g(P)\hat{f}(P) \qquad (P \in \Gamma).$$

We use the notation and facts presented above in what follows.

PROPOSITION 7. *Let B be a C^*-algebra with Hausdorff structure space Γ. Assume $\rho \in C_0(\Gamma)$ with $\rho(\gamma) \geqslant 0$ for all $\gamma \in \Gamma$. There exists $f \in B$ such that*

$$\|\hat{f}(\gamma)\| = \rho(\gamma) \qquad (\gamma \in \Gamma).$$

PROOF. For each $n \geqslant 1$ let K_n be the compact set

$$K_n = \{\gamma \in \Gamma: \rho(\gamma) \geqslant \tfrac{1}{n}\}.$$

Let $V = \{\gamma \in \Gamma: \rho(\gamma) > 0\} = \bigcup_{n=1}^\infty K_n$. Fix n. Since for each $\gamma \in K_n$ there exists $w \in B$ such that $\|\hat{w}(\gamma)\| \neq 0$ and K_n is compact, we can construct an element $v_n \in B$ such that $\|\hat{v}_n(\gamma)\| \neq 0$ for all $\gamma \in K_n$. Let

$$u = \sum_{k=1}^{\infty} \left(\frac{1}{2}\right)^k (\|v_k\|_\infty)^{-2} v_k^* v_k.$$

It is not difficult to see that $\|\hat{u}(\gamma)\| \neq 0$ for all $\gamma \in V$. For each n choose $g_n \in C_0(\Gamma)$ such that

$$g_n(\gamma) = 1, \quad \gamma \in K_n, \qquad g_n(\gamma) = 0, \quad \gamma \notin K_{n+1},$$
$$0 \leqslant g_n(\gamma) \leqslant 1, \quad \gamma \in \Gamma.$$

Let $f_n \in B$ take the values

$$\hat{f}_n(\gamma) = g_n(\gamma)\|\hat{u}(\gamma)\|^{-1}\rho(\gamma)\hat{u}(\gamma) \qquad (\gamma \in \Gamma).$$

Note that $\|\hat{f}_n(\gamma)\| = \rho(\gamma)$ for $\gamma \in K_n$, and $\|\hat{f}_k(\gamma)\| \leqslant \rho(\gamma) < 1/n$ for $\gamma \notin K_n$ and $k \geqslant 1$. If $n \geqslant m$, then

$$\hat{f}_n(\gamma) - \hat{f}_m(\gamma) = 0 \quad \text{for } \gamma \in K_m,$$

$$\|\hat{f}_n(\gamma) - \hat{f}_m(\gamma)\| \leqslant \|\hat{f}_n(\gamma)\| + \|\hat{f}_m(\gamma)\| \leqslant 1/m + 1/m \quad \text{for } \gamma \notin K_m.$$

Therefore $\{f_n\}$ is a Cauchy sequence in B. Set $f = \lim_n f_n \in B$. Then

$$\|\hat{f}(\gamma)\| = \lim\|\hat{f}_n(\gamma)\| = \rho(\gamma)$$

for all $\gamma \in \Gamma$ since for each m, $\|\hat{f}_n\| = \rho(\gamma)$ for $n \geqslant m$, $\gamma \in K_m$.

THEOREM 8. *Let A be a Banach *-algebra with a bounded left (or right) approximate identity. Let φ be a *-homomorphism of A into a dense subalgebra of a C^*-algebra B with Hausdorff structure space Γ. If $\rho \in C_0(\Gamma)$ and $\rho(\sigma) > 0$ for all $\sigma \in \Gamma$, then there exists $f \in A$ such that*

$$\|\widehat{\varphi(f)}(\sigma)\| \geqslant \rho(\sigma) \qquad (\sigma \in \Gamma).$$

PROOF. By Proposition 7 there exists $g \in B$ such that

$$\|\hat{g}(\sigma)\| = \rho(\sigma) \qquad (\sigma \in \Gamma).$$

We have $\varphi(A) = \varphi(A^2) \subset \varphi(A)B$. Since $\varphi(A)$ is dense in B, by the Cohen Factorization Theorem

$$\varphi(A)B = \text{CLS}(\varphi(A)B) = B.$$

Choose $f_1 \in A$ and $g_1 \in B$ such that $\varphi(f_1)g_1 = g$. Let $f = \|g_1\|f_1$. Then for all $\sigma \in \Gamma$,

$$\|\widehat{\varphi(f)}(\sigma)\| = \|g_1\| \, \|\widehat{\varphi(f_1)}(\sigma)\| \geqslant \|\hat{g}(\sigma)\| = \rho(\sigma).$$

There are several classes of groups G (beside abelian or compact groups) for which $C^*(G)$, the C^*-group algebra of G, has Hausdorff structure space; see the section of **[8]** concerning the class $[T_2]$. When this is the case Theorem 8 applies where $A = L^1(G)$ and φ is the natural embedding of A into $B = C^*(G)$.

REFERENCES

1. L. Baggett, *Hilbert-Schmidt representations of groups*, Proc. Amer. Math. Soc. **21** (1969), 502–506.

2. J. Dixmier, *Les C*-algèbres et leurs représentations*, Gauthier-Villars, Paris, 1964.

3. N. Dunford and J. T. Schwartz, *Linear operators*, Part II, Interscience, New York, 1963.

4. G. A. Elliott and D. Olesen, *A simple proof of the Dauns-Hofmann Theorem*, Math. Scand. **33** (1974), 231–234.

5. E. Hewitt and K. Ross, *Abstract harmonic analysis*, Vol. II, Springer-Verlag, Berlin and New York, 1970.

6. I. Kaplansky, *The structure of certain operator algebras*, Trans. Amer. Math. Soc. **70** (1951), 219–255.

7. R. Mosak, *Central functions in group algebras*, Proc. Amer. Math. Soc. **29** (1971), 613–616.

8. T. W. Palmer, *Classes of nonabelian, noncompact, locally compact groups*, Rocky Mountain J. Math. **8** (1978), 683–741.

9. H. Reiter, *Classical harmonic analysis and locally compact groups*, Oxford Univ. Press, London, 1968.

10. G. Warner, *Harmonic analysis on semi-simple Lie groups*. I, Springer-Verlag, Berlin and New York, 1972.

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