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A NOTE ON SINGULAR INTEGRALS

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ABSTRACT. In this article we discuss what happens when we consider a convolution operator whose kernel is a Calderón-Zygmund kernel multiplied by a bounded radial function. Some generalizations are obtained.

The purpose of this note is to obtain the following results.

THEOREM. *Let* $\Omega(x)/|x|^n = K(x)$ *be a Calderón-Zygmund kernel in* R^n, $n \geqslant 2$ (*i.e., let* $\Omega(x)$ *be homogeneous of degree* 0, $\int_{S^{n-1}} \Omega(x)d\sigma(x) = 0$, *and suppose* Ω *satisfies a Lipschitz condition of positive order on* S^{n-1}). *Let* $h(x)$ *be any bounded radial function, and put* $H(x) = h(x) \cdot K(x)$. *Then the convolution operator* $T(f) = f * H$ *is bounded on* $L^p(R^n)$, *if* $1 < p < \infty$.

Notice that such singular integrals cannot be treated by use of the classical arguments, since, in general, nothing can be said about integrals like

$$\int_{|x|>2|y|} |H(x+y) - H(x)|dx,$$

because H could be so rough.

Nevertheless, we shall prove the somewhat stronger theorem below, whose theme is that smoothness in the radial direction for a convolution kernel is unnecessary in order to have the boundedness of the corresponding operator.

THEOREM. (a) *Suppose that for each* $r > 0$, *we are given a function* Ω_r *defined on* S^{n-1} *in such a way that the family* $\{\Omega_r\}$ *is uniformly in the Dini class* (*i.e., if* $\omega^*(\delta) = \sup\{|\Omega_r(x) - \Omega_r(y)|: \ x, y \in S^{n-1}, \ |x-y| < \delta, \ r > 0\}$, *then* $\int_0^1 \omega^*(\delta)d\delta/\delta < \infty$) *and also* $\int_{S^{n-1}} \Omega_r(x)d\sigma(x) = 0$. *Let*

$$H(x) = \Omega_{|x|}(x/|x|)/|x|^n.$$

Then $\|H * f\|_2 \leqslant C\|f\|_2$.

(b) *Suppose in part* (a) *we replace the Dini class by a Lipschitz class of some positive order. Then* $\|H * f\|_p \leqslant C_p\|f\|_p$, $1 < p < \infty$.

At this point, we would like to remark that in this work, we were very much motivated by the work of E. M. Stein on maximal spherical averages (see [3]).

PROOF OF (a). We estimate the Fourier transform of H:

$$\hat{H}(\xi) = \int_{R^n} \frac{\Omega_{|x|}(x/|x|)}{|x|^n} e^{-i\xi\cdot x}\, dx = \int_0^\infty \left[\Omega_r(x')d\sigma(x')\right]^\wedge(r\xi)\frac{dr}{r}.$$

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Let us get an estimate for $[\Omega_r(x')d\sigma(x')]^\wedge(\xi)$. By the spherical symmetry we may assume $\xi = (|\xi|, 0, 0, \dots, 0)$.

Then if $S_t = \{(x_1, x_2, \dots, x_n) \in S^{n-1}: x_1 = t\}$, we have

$$[\Omega_r(x')d\sigma(x')]^\wedge(\xi) = \int_0^\pi e^{i|\xi|\cos\theta} \int_{S_{\cos\theta}} \Omega_r(t)\, d\sigma_{n-2}^\theta(t)\, d\theta$$

where $d\sigma_{n-2}^\theta$ is the unit of surface area on the sphere $S_{\cos\theta}$. Changing variables this becomes

$$\int_{-1}^{+1} e^{i|\xi|s} \int_{S_s} \Omega_r(t)\, d\sigma_{n-2}^s(t) \frac{1}{\sqrt{1-s^2}}\, ds$$

$$= \int_{-1}^{+1} e^{i|\xi|s} (1-s^2)^{(n-3)/2} \int_{S^{n-2}} \Omega(s, \sqrt{1-s^2}\, t')\, d\sigma_{n-2}(t')\, ds$$

($d\sigma_{n-2}^s$ the unit of area on S_s) where we have changed variables so that the inner integral is taken over a unit sphere, S^{n-2}. If we set

$$I(s) = \int_{S_s} \Omega(s, \sqrt{1-s^2}\, t')\, d\sigma_{n-2}(t')$$

then[2]

$$\left| \int_{-1}^{+1} e^{i|\xi|s} (1-s^2)^{(n-3)/2} I(s)\, ds \right|$$

$$< \int_{1-10/|\xi|<|s|<1} (1-s^2)^{(n-3)/2} ds$$

$$+ \int_{|s|<1-10/|\xi|} \left| \left[1 - \left(s + \frac{\pi}{|\xi|}\right)^2\right]^{(n-3)/2} I\left(s + \frac{\pi}{|\xi|}\right) - (1-s^2)^{(n-3)/2} I(s) \right| ds$$

$$< \frac{C}{|\xi|^{(n-1)/2}} + \int_{|s|<1-10/|\xi|} \left| \left[1 - \left(s + \frac{1}{|\xi|}\right)^2\right]^{(n-3)/2} - (1-s^2)^{(n-3)/2} \right| \left| I\left(s + \frac{1}{|\xi|}\right) \right| ds$$

$$+ \int_{|s|<1-10/|\xi|} (1-s^2)^{(n-3)/2} \left| I\left(s + \frac{1}{|\xi|}\right) - I(s) \right| ds$$

$$< \frac{C}{|\xi|^{1/2}} + \int_{|s|<1-10/|\xi|} (1-s^2)^{(n-3)/2} \left| I\left(s + \frac{1}{|\xi|}\right) - I(s) \right| ds.$$

[2]We warn the reader that in some inequalities we have ignored unimportant multiplicative constants.

Now

$$\left| I\left(s + \frac{1}{|\xi|}\right) - I(s) \right|$$

$$\leqslant \sup_{t' \in S^{n-2}} \left| \Omega_r\left(s + \frac{1}{|\xi|}, \sqrt{1 - \left(s + \frac{1}{|\xi|}\right)^2}\, t'\right) - \Omega_r\left(s, \sqrt{1 - s^2}\, t'\right) \right|$$

$$\leqslant \omega^*\left(\frac{1}{|\xi|^{1/2}}\right).$$

So putting this together we see that

$$|\Omega_r(x')d\sigma(x')^{\wedge}(\xi)| \leqslant C\left[\frac{1}{|\xi|^{1/2}} + \omega^*\left(\frac{1}{|\xi|^{1/2}}\right)\right].$$

This is the estimate required for $|\xi| \geqslant 1$. For $|\xi| \leqslant 1$, $|[\Omega_r(x')d\sigma(x')]^{\wedge}(\xi)| \leqslant C|\xi|$, since $\widehat{\Omega_r d\sigma}(0) = 0$ ($\int_{S^{n-1}} \Omega_r d\sigma = 0$) and $[\Omega_r d\sigma]^{\wedge}$ is smooth, being the Fourier transform of a compactly supported measure. Then

$$|\hat{H}(\xi)| \leqslant \int_0^\infty J(r|\xi|)\, \frac{dr}{r} = \int_0^\infty J(r)\, \frac{dr}{r}$$

where

$$J(r) = \begin{cases} c\left[\dfrac{1}{r^{1/2}} + \omega^*\left(\dfrac{1}{r^{1/2}}\right)\right], & r \geqslant 1, \\ cr, & r \leqslant 1. \end{cases}$$

Of course $\int_0^\infty J(r)dr/r < \infty$, since, for example,

$$\int_1^\infty \omega^*\left(\frac{1}{r^{1/2}}\right) \frac{dr}{r} \sim \int_0^1 \omega^*(\delta)\, \frac{d\delta}{\delta}.$$

This finishes the proof of (a).

Now, in order to prove an L^p estimate $1 < p < \infty$, we shall view the singular integral as roughly speaking, a singular integral along a curve, using the methods of Nagel, Rivière and Wainger [2].

Consider the operators T_α, for α complex, defined by $T_\alpha f = I_\alpha(f * H(x)/|x|^\alpha)$, where $\widehat{I_\alpha(f)}(\xi) = |\xi|^{-\alpha}\hat{f}(\xi)$. We shall first observe that if $-\eta < \operatorname{Re}\alpha < +\eta$, T_α is bounded on $L^2(R^n)$. Then for $0 < \operatorname{Re}\alpha < \eta$, if we can show that T_α is bounded on all $L^p(R^n)$, $1 < p < \infty$, by Stein's interpolation theorem, that T_0 is bounded on all the classes L^p.

To see that T_α is bounded on L^2 wherever $|\operatorname{Re}\alpha|$ is small, we use the method of part (a):

$$T_\alpha f = f * H_\alpha,$$

where

$$\hat{H}_\alpha(\xi) = \int_0^\infty [\Omega_r(x')d\sigma(x')]^{\wedge}(r\xi)\, \frac{dr}{r^{1+\alpha}} \cdot |\xi|^{-\alpha},$$

and the type of reasoning used in part (a) shows that $\hat{H}_\alpha(\xi)$ is bounded if $|\mathrm{Re}\,\alpha|$ is small. (We use here the fact that Ω_r is Lipschitz.)

Finally if $\mathrm{Re}\,\alpha > 0$ it follows that

$$\int_{|x|>2|y|} |H_\alpha(x+y) - H_\alpha(x)|dx \leqslant C_\alpha, \tag{$\sim$}$$

where C_α does not depend on y. By the methods of Calderón and Zygmund [1], T is bounded on all L^p, and we are done.

To see $(\sim)$, we shall prove that

$$\int_{|x|>2} |H_\alpha(x+y) - H_\alpha(x)|dx \leqslant C_\alpha, \tag{$\approx$}$$

assuming $|y| = 1$.

Then we observe that $H_\alpha(\delta x) = \delta^{-n}\tilde{H}_\alpha(x)$ where $\tilde{H}_\alpha$ differs from H_α only in the way the family Ω_r is parameterized; ω^* is invariant in the passage from H_α to $\tilde{H}_\alpha$ so that it is indeed enough to prove $(\approx)$:

$$H_\alpha = c_\alpha \cdot \frac{1}{|t|^{n-\alpha}} * \frac{\Omega_{|t|}(t')}{|t|^{n+\alpha}}.$$

Let

$$H_\alpha^1 = c_\alpha \frac{1}{|t|^{n-\alpha}} * X_{|t|<1}(t)\frac{\Omega_{|t|}(t')}{|t|^{n+\alpha}}, \qquad H_\alpha^2 = c_\alpha \frac{1}{|t|^{n-\alpha}} * X_{|t|>1}\frac{\Omega_{|t|}(t')}{|t|^{n+\alpha}}$$

and assume $\mathrm{Re}\,\alpha > 0$.

We have, for $|x| > 2$,

$$|H_\alpha^1(x)| = \left| c_\alpha \int_{|t|<1} \left[\frac{1}{|x-t|^{n-\alpha}} - \frac{1}{|x|^{n-\alpha}} \right] \frac{\Omega_{|t|}(t')}{|t|^{n+\alpha}}\, dt \right|$$

$$\leqslant \int_{|t|<1} \frac{|t|}{|x|^{n-\mathrm{Re}\,\alpha+1}} \cdot \frac{1}{|t|^{n+\mathrm{Re}\,\alpha}}\, dt \leqslant \frac{C_\alpha}{|x|^{n-\mathrm{Re}\,\alpha+1}}$$

and

$$\int_{|x|>2} |H_\alpha^1(x)|dx \leqslant C_\alpha.$$

As for H_α^2, we have

$$\int_{|x|>2} |H_\alpha^2(x+y) - H_\alpha^2(x)|dx \leqslant \left\| \left[\frac{1}{|x+y|^{n-\alpha}} - \frac{1}{|x|^{n-\alpha}} \right] * \frac{\Omega_{|x|}(x')}{|x|^{n+\alpha}} \right\|_1$$

$$\leqslant \left\| \frac{1}{|x+y|^{n-\alpha}} - \frac{1}{|x|^{n-\alpha}} \right\|_{L^1(|x|>1)} \left\| \frac{\Omega_{|x|}(x')}{|x|^{n+\alpha}} \right\|_{L^1(|x|>1)} < \infty.$$

This concludes the proof of (b).

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