

GEOMETRIC ESTIMATION OF THE SOLUTION TO
 $x + Tx = 0$ FOR UNBOUNDED DENSELY
DEFINED MONOTONE OPERATOR T IN HILBERT SPACE

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ABSTRACT. In recent papers R. Bruck and J. C. Dunn have given convergent schemes for approximating the solution p of $x + Tx = f$ for T a monotone mapping on a Hilbert space, with T locally bounded. The present paper derives a geometric fact and uses this in a direct manner to give a scheme applicable to densely defined T . The scheme is computable with decreasing error estimates without any assumptions of boundedness. The convergence of the scheme to the solution p is proven under the weak condition that $\|x_n + Tx_n\|$ grow no faster than $n^{1/2}$.

Introduction. Let $T: D \subseteq H \rightarrow H$, H a Hilbert space, satisfy for some $c < 1$ and all $x, y \in D$

$$\operatorname{Re}\langle Tx - Ty, x - y \rangle \geq -c\|x - y\|^2. \quad (1)$$

For $c \leq 0$ Vainberg [6], Zarantonello [9], Minty [4], Browder [1], and Petryshyn [5] have made contributions to constructing a solution p of $x + Tx = 0$, with some additional continuity assumptions on T . For $c \leq 0$ and under only the mild condition D open, Bruck [2] has given a convergent iteration scheme, provided that the initial guess was sufficiently close to p . The "closeness" required was existential since it depended on the constant of local boundedness for T at p which resulted from D being open. Recently Dunn [3] has extended this result to $c < 1$ with the assumptions that D is convex, $T: D \rightarrow D$, and T is bounded. Both Bruck and Dunn gave rates of convergence $\|x_n - p\| = O(n^{-1/2})$.

The present paper will:

- (i) provide a geometric description of condition (1) immediately yielding one-step error estimates for p (without *a priori* existence assumption).
- (ii) use this description to develop an iteration process which has strictly decreasing error estimates (without *a priori* boundedness assumptions).
- (iii) demonstrate that the iterates must necessarily converge to p under either of two conditions:
 - (a) the image of a certain ball is bounded, in which case the error estimates are $O(n^{-1/2})$,
 - (b) a certain sequence of displacements grows no faster than $n^{1/2}$.

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(iv) show that this scheme is stable under perturbations and hence is applicable to operators with dense domain.

Additionally these results hold for multivalued mapping as do those of Bruck and Dunn.

The method. For ease of exposition, we consider an equivalent problem by replacing T by $-T$ in (1) giving

$$\operatorname{Re}\langle Tx - Ty, x - y \rangle \leq c\|x - y\|^2 \tag{2}$$

for all $x, y \in D$, and assume that p is the (necessarily unique) solution to $Tx = x$.

We note that $y \in H$ cannot equal p if for any $x \in D$

$$\operatorname{Re}\langle Tx - y, x - y \rangle > c\|x - y\|^2. \tag{3}$$

Let $x \in D$. If $x \neq p$ define $q = \lambda x + (1 - \lambda)Tx$, where

$$\lambda = (1 - 2c)(2 - 2c)^{-1},$$

and let $\rho = \|x - Tx\|(2 - 2c)^{-1}$.

The following theorem gives the location of the fixed point p relative to the points x and Tx , namely, p must lie in the closed ball $B(q, \rho)$.

THEOREM A. *If $y \notin B(q, \rho)$, then $y \neq p$.*

PROOF. $\rho < \|y - q\|$.

$$\begin{aligned} \|x - Tx\|^2 &= 4(1 - c)^2\rho^2 < 4(1 - c)^2\|y - q\|^2 \\ &= (1 - 2c)^2\|x - y\|^2 + \|Tx - y\|^2 + 2(1 - 2c)\operatorname{Re}\langle Tx - y, x - y \rangle \\ &= \{\|x - y\|^2 + \|Tx - y\|^2 - 2\operatorname{Re}\langle Tx - y, x - y \rangle\} \\ &\quad + \{2\operatorname{Re}\langle Tx - y, x - y \rangle + 2(1 - 2c)\operatorname{Re}\langle Tx - y, x - y \rangle \\ &\quad\quad\quad + (4c^2 - 4c)\|x - y\|^2\} \\ &= \|x - Tx\|^2 + (4 - 4c)\{\operatorname{Re}\langle Tx - y, x - y \rangle - c\|x - y\|^2\}. \end{aligned}$$

Subtracting $\|x - Tx\|^2$ from both sides yields

$$0 < (4 - 4c)\{\operatorname{Re}\langle Tx - y, x - y \rangle - c\|x - y\|^2\}.$$

Since $c < 1$, (3) must hold concluding the theorem.

Let $R_x = B(q, \rho)$, the set in which p must be contained. The set R_x is optimal (best possible) set in which p can be guaranteed to lie, that is, given only x, Tx and c satisfying the condition (2) the fixed point p could possibly be any point of R_x (see [7]). The argument will be omitted.

Note also that $x \in \partial R_x$ since $\|x - q\| = \rho$.

The iteration will involve only the intersection of such balls. Let us for the moment assume $D = H$, and choose $x_0 \in D$. Let $x_1 = q_0$ and $r_1 = \rho_0$, which are the q and ρ obtained from R_{x_0} . Let us assume that x_i and r_i have been

constructed, with q_i and ρ_i obtained from R_{x_i} . To construct x_{i+1} and r_{i+1} we consider three situations:

Case 0. $\rho_i = 0$. x_i is the fixed point.

Case 1. $\rho_i < 2^{-1/2}r_i$. Let $r_{i+1} = \rho_i$ and $x_{i+1} = q_i$.

Case 2. $\rho_i > 2^{-1/2}r_i$. We want to determine the smallest ball $B(x_{i+1}, r_{i+1})$ which will contain $I = B(x_i, r_i) \cap R_{x_i}$. It suffices to choose $y \in I$ and restrict our consideration to the real two dimensional plane generated by $\{y, x_i, Tx_i\}$.

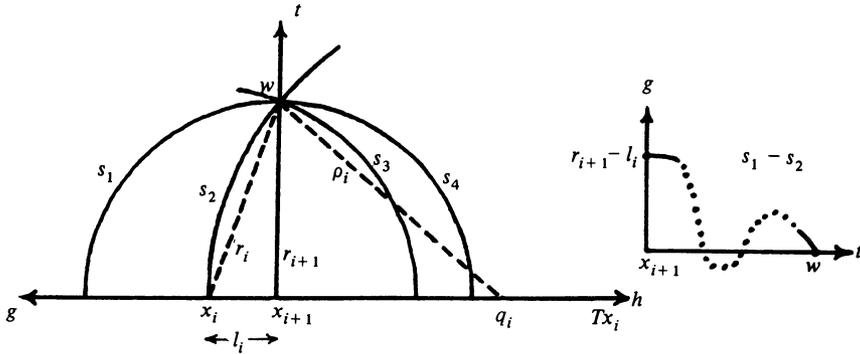


FIGURE 1

Now (Figure 1) x_{i+1} and r_{i+1} must satisfy:

$$\begin{aligned} r_i^2 &= r_{i+1}^2 + l_i^2, & r_{i+1}^2 + (\rho_i - l_i)^2 &= \rho_i^2, \\ x_{i+1} &= (1 - \gamma)x_i + \gamma Tx_i, \end{aligned}$$

with $\gamma > 0$ chosen so that

$$\|x_i - x_{i+1}\| = l_i.$$

The first two of these imply $l_i = r_i^2/2\rho_i$. $\|x_i - x_{i+1}\| = \gamma\|x_i - Tx_i\| = l_i$ implies

$$\gamma = \frac{r_i^2}{2\rho_i\|x_i - Tx_i\|} = \frac{r_i^2}{4\rho_i^2(1 - c)}.$$

In order to verify that $B(x_{i+1}, r_{i+1})$ contains $B(x_i, r_i) \cap R_{x_i}$ in this Case 2 construction it suffices to show that the arc s_2 on ∂B_{x_i} and the arc s_3 on $\partial B(x_i, r_i)$ lie in $B(x_{i+1}, r_{i+1})$ (see Figure 1). Hence it suffices to show that the function $g(t) = s_1 - s_2$ is ≥ 0 on the interval $[x_{i+1}, w]$ (see detail on Figure 1). Now $g(x_{i+1}) = r_{i+1} - l_i > 0$ for Case 2. Combining $g(x_{i+1}) > 0$, $g(w) = 0$ and $g'(x_{i+1}) = 0$ with the fact that $g(t)$ is quadratic implies $g(t) \geq 0$ on $[x_{i+1}, w]$. Similarly $l_i + r_{i+1} > r_i$ implies that $h(t) = s_4 - s_3 \geq 0$ on the interval $[x_{i+1}, w]$. This completes the containment argument. Hence $p \in B(x_{i+1}, r_{i+1})$.

Moreover, since $r_{i+1}^2 = r_i^2[1 - r_i^2(2\rho_i)^{-2}]$ and $0 < r_i^2(2\rho_i)^{-2} < 1$, we get that $r_{i+1} < r_i$. Therefore we have the following theorem.

THEOREM B. *Let T satisfy (2) with $D = H$ and fixed point p . Let $x_0 \in D$, and $r_1 = \rho_0$, $x_1 = q_0$ from R_{x_0} . For $i \geq 0$, define $x_{i+1} = q_i$ and $r_{i+1} = \rho_i$ if $\rho_i \leq 2^{-1/2}r_i$; otherwise define*

$$x_{i+1} = (1 - \gamma_i)x_i + \gamma_iTx_i \quad \text{where } \gamma_i = r_i^2[4\rho_i^2(1 - c)]^{-1}$$

and

$$r_{i+1} = r_i[1 - r_i^2(2\rho_i)^{-2}]^{1/2}.$$

Then $r_n > r_{n+1}$ and $p \in B(x_n, r_n)$ for every n .

Before discussing the conditions which would guarantee $r_n \rightarrow 0$ we comment that given any ball B such that $p \in B$, the n th iterate can be assured to lie in B by replacing (if necessary) $B(x_n, r_n)$ with $B(x'_n, r'_n)$ which is the smallest ball containing the intersection of B and $B(x_n, r_n)$ constructed in a manner similar to above. $x'_n \in B$, $r'_n < r_n$ and $p \in B(x'_n, r'_n)$. Consequently, this allows a ball instead of all of H for the domain of T , or say allows the choosing of all iterates $x_{n+k} \in B(x_n, r_n)$ for all $k \geq 0$ where here we mean x_i to stand for either x_i or x'_i if necessary.

We now discuss conditions which insure $r_n \rightarrow 0$. If an infinite number of iterates are formed as in Case 1 above, then given any n there exist an $m \geq n$ and constant $K < 1$ such that $r_m \leq Kr_1(2^{-1/2})^n$; hence $r_n \rightarrow 0$ without additional conditions. Otherwise only a finite number of iterates are formed as in Case 1. Hence there exists N such that for all $n \geq N$

$$r_n^2 \leq r_N^2 \prod_{i=N+1}^{n-1} \left[1 - \left(\frac{r_i}{2\rho_i} \right)^2 \right]. \tag{4}$$

Let $a_i = r_i^2(2\rho_i)^{-2}$, note $0 < a_i < 1$ and consider $\prod_{i=N+1}^{\infty} (1 - a_i)$. This infinite product either converges to $t > 0$ or diverges to zero. If it diverges to zero, (4) implies $r_n \rightarrow 0$. If it converges to t , then $\sum_{i=N+1}^{\infty} a_i$ converges which implies $a_n \rightarrow 0$. By the definition of a_n , it suffices to provide conditions which imply ρ_n bounded above in order to conclude $r_n \rightarrow 0$. If for some N , $B(x_N, r_N) \subseteq D$, and if image of this ball under T is bounded, and if we keep the subsequent iterates in this ball, we can conclude ρ_n is bounded.

THEOREM C. $r_n \rightarrow 0$, that is, $x_n \rightarrow p$, as $n \rightarrow \infty$ provided that T satisfies (2) and for some N

- (c1) $B(x_N, r_N) \subseteq D$,
- (c2) $T(B(x_N, r_N))$ is bounded,
- (c3) x_{N+k} are kept in $B(x_N, r_N)$ for all k .

(c3) is no additional condition, merely a constructional note (use x'_{N+k} if necessary). (c1) is included only to insure that all subsequent iterates x_i or x'_i lie in the domain of T .

We now use the inherent stability of this method to allow the weakening of

(c1) and (c2) in order to make the method available to operators T with dense domain.

Suppose for some N

(c4) $D \cap B(x_N, r_N)$ is dense in $B(x_N, r_N)$,

(c5) $T(D \cap B(x_N, r_N))$ is bounded.

Assume that for some $n \geq N$, $B(x_n, r_n)$ has been constructed with $x_n \in D$. Obtain ρ_n and q_n from R_{x_n} . If the usual construction of x_{n+1} (or x'_{n+1} if necessary) yields $x_{n+1} \notin D$, we need to construct an $x''_{n+1} \in D$ and r''_{n+1} so as to keep $p \in B(x''_{n+1}, r''_{n+1})$ and have our convergence arguments still hold.

If x_{n+1} and r_{n+1} were formed as in Case 1 ($\rho_n \leq 2^{-1/2}r_n$), choose $x''_{n+1} \in D$ such that $\|x''_{n+1} - x_{n+1}\| \leq 0.09r_n$ and choose $r''_{n+1} = 0.8r_n$. Noting $2^{-1/2} < 0.71$, we have $B(x''_{n+1}, r''_{n+1}) \supseteq B(x_{n+1}, r_{n+1})$.

If x_{n+1} and r_{n+1} were formed as in Case 2 ($\rho_n > 2^{-1/2}r_n$), let

$$f = 4^{-1}r_n^4[\rho_n^{-2} - (\rho_n + 1)^{-2}] > 0.$$

Now choose $r''_{n+1} > r_{n+1}$ such that $(r''_{n+1})^2 \leq r_{n+1}^2 + f$, and choose $x''_{n+1} \in D$ such that $\|x''_{n+1} - x_{n+1}\| \leq r''_{n+1} - r_{n+1}$.

$$(r''_{n+1})^2 \leq r_{n+1}^2 + f \leq r_n^2[1 - r_n^2(2\rho_n)^{-2}] + f \leq r_n^2[1 - r_n^2(2(\rho_n + 1))^{-2}].$$

Hence we have $r''_{n+1} < r_n$ and $B(x''_{n+1}, r''_{n+1}) \supseteq B(x_{n+1}, r_{n+1})$. Therefore for any n we have (suppressing any prime marks)

$$r_{n+1}^2 \leq r_1^2 \prod_{i=2}^n \delta_i$$

where either $\delta_i = (0.8)^2$ or $\delta_i = 1 - r_i^2(2\rho_i + 2)^{-2}$.

The convergence argument proceeds as before.

THEOREM D. $r_n \rightarrow 0$, that is, $x_n \rightarrow p$, as $n \rightarrow \infty$ provided that T satisfies (2), and for some N (c4) and (c5) hold and the iterates x_{N+k} are kept in the set $D \cap B(x_N, r_N)$ as prescribed.

THEOREM E. Assuming $\{\rho_n\}$ bounded by M , convergence is $O(n^{-1/2})$.

PROOF. First, at each step maximum occurs when $\rho_i = M$ for all i and Case 2 construction, yielding

$$r_1 = M, \quad r_{N+1}^2 = r_n^2 - r_n^4(2M)^{-2} \quad \text{for } n > 1.$$

Hence it suffices to show $\{a_n\}$ converges to zero $O(n^{-1})$, where

$$a_1 = \frac{1}{4}L^2, \quad a_{n+1} = a_n - a_n^2L^{-2} \quad \text{for } n > 1, \quad L = 2M.$$

LEMMA. $L^2n^{-1} \geq a_n, n \geq 1$.

PROOF (BY INDUCTION). The inductive step first: If $L^2n^{-1} \geq a_n$, show $L^2(n+1)^{-1} \geq a_{n+1}$. Let $L^2n^{-1} = a_n + e, e \geq 0$. It suffices to show:

$$\begin{aligned}
 L^2(n + 1)^{-1} &\geq a_n - a_n^2 L^{-2} = (L^2 n^{-1} - e) - L^{-2}(L^2 n^{-1} - e)^2, \\
 L^2(n + 1)^{-1} &\geq L^2 n^{-1} - e - L^{-2}(L^4 n^{-2} - 2eL^2 n^{-1} + e^2), \\
 L^2 &\geq L^2(n + 1)n^{-1} - e(n + 1) - L^2(n + 1)n^{-2} \\
 &\quad + 2e(n + 1)n^{-1} - e^2 L^{-2}(n + 1), \\
 0 &\geq L^2 n^{-1} - e(n + 1) - L^2 n^{-1} - L^2 n^{-2} + 2e + 2en^{-1} - e^2 L^{-2}(n + 1), \\
 0 &\geq -e((n + 1) - 2 - 2n^{-1}) - L^2 n^{-2} - e^2 L^{-2}(n + 1).
 \end{aligned}$$

Since $0 \geq -e((n - 1) - 2n^{-1})$ for $n \geq 2$, the theorem is complete once $L^2 n^{-1} \geq a_n$ is verified for $n = 2$, which is true by direct computation. \square

In order to remove the boundedness condition and still conclude that $r_n \rightarrow 0$, we re-examine the discussion prior to Theorem C. The assumption of boundedness was used only at the point where $\sum a_n$ was convergent, with $a_n = r_n^2(2\rho_n)^{-2}$ and $\rho_n = \|x_n - Tx_n\|$.

Since $\sum a_n$ converges then there exists a subsequence a_{n_k} such that $n_k a_{n_k} \rightarrow 0, k \rightarrow \infty$.

$$r_{n_k}^2 = 4n_k a_{n_k} (\rho_{n_k}^2 n_k^{-1}).$$

So it would suffice to imply $\rho_{n_k}^2 n_k^{-1}$ is bounded. Hence, instead of assuming ρ_n bounded we need only assume that ρ_n grows no faster than $n^{1/2}$ in order to conclude $r_{n_k} \rightarrow 0$.

Since r_n is monotone we would have $r_n \rightarrow 0$.

THEOREM F. $r_n \rightarrow 0$ provided that T satisfies (2) and for some N (c1) and (c3) hold and there exists $M > 0$ such that

$$\rho_n \leq Mn^{1/2} \text{ for all } n \geq N.$$

Similarly Theorem D can be modified to apply to densely defined unbounded mappings T .

Comments. Re-inspection of the above theorems shows that *a priori* existence of the fixed point is not required since Theorem A can be rephrased "if a fixed point exists, it must lie in $B(q, \rho)$." In fact, if for any $y, z \in D, R_y \cap R_z = \emptyset$, then we conclude T has no fixed point.

Our error estimates are explicitly determined, involving only data known at any given step, not the overall bound for ρ_n (that is, for T). The iteration construction, with error estimates, can proceed without the overall bound on ρ_n and is stable in the sense exploited in Theorem D. Perhaps more important than $\{r_n\}$ being strictly monotone decreasing is the fact that the value of r_n (as well as the coefficient γ_n) is not chosen prior to the start of construction of iterates, allowing the occurrence of a small value of $\|x_n - Tx_n\|$ to be capitalized upon during the construction of x_{n+1} .

Numerically the estimates of the Lemma to Theorem E appear to be best possible *a priori* estimates for this scheme. Specifically, when compared to 10,000 terms $na_n \sim L^2$.

If in Dunn's example [3, note 2] we let $T(0) = \{0\}$ and let T_1 be any single-valued selection, then T_1 has only one fixed point $p = 0$, T_1 is not continuous at p and

(i) there exists $\eta > 0$ such that for every $x \neq p$, $\|T_1x - x\| \geq \eta$,

(ii) for every $x \neq p$, $\|p - T_1x\| = 1$.

(i) shows the futility of looking for fixed points by means of approximate fixed points, while (ii) shows why, in the iterations of Bruck and Dunn, the scheme of coefficients of Tx_i must vanish in the limit. A final observation is that at $c = 1$ our theorems fail since our geometric set R_x (Theorem A) ceases to be a bounded set (ball)—this seems to confirm intuitively the coincidence that Dunn's analysis failed at $c = 1$ for technical reasons.

We offer several conjectures:

(i) An example could be constructed with $\{\rho_n\}$ increasing rapidly enough to have $r_n \rightarrow 0$ for the iteration scheme of Theorem B.

(ii) If $\{r_n\}$ is not decreasing rapidly enough (that is, $\{\rho_n\}$ is increasing rapidly)—perhaps restart the method at some other point y_0 , iterate a few steps and intersect the last ball of each scheme to yield a smaller ball than either.

Added Note. I would like to thank the referee for informing me that the Lemma to Theorem E follows directly from [3, Lemma 2], and for indicating the existence of [10] which deals with locally bounded monotone mappings.

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