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INFINITE DIMENSIONAL L-SPACES DO NOT HAVE PREDUALS OF ALL ORDERS[1]

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ABSTRACT. It is shown that if E is an infinite dimensional Banach space with first dual E', second dual E'', and nth dual $E^{[n]}$ and if $E^{[n]}$ is either an L- or M-space all duals are either L- or M-spaces except possibly E which could be a Lindenstrauss space. If E is an L- or M-space there is an integer $n(E)$ so that if $m > n(E)$ there is no Banach space F with $E = F^{[m]}$. The linear isomorphic analogues to these isometric results are also established. In particular if E is an $\mathfrak{L}_1$ or $\mathfrak{L}_\infty$ space there is an integer $\bar{n}(E)$ so that E is not linearly isomorphic to $F^{[m]}$ for any Banach space F when $m > \bar{n}(E)$.

Introduction. Consider a sequence $\{E_i\colon i = 1, \ldots, n\}$ of Banach spaces with $E'_{j+1} = E_j$ for $j = 1, \ldots, n-1$. We call E_2 a predual of E_1 and E_2 a pre-bidual of E_1. In general we say that E_{j+1} is a predual of E_1 of order j and denote this fact by the equation $E_{j+1}^{[j]} = E_1$. Is it possible that there exists, for a given Banach space E, an infinite sequence $\{E_j\colon j \in N\}$ of Banach spaces with $E_1 = E$ and with E_j the dual of E_{j+1} for all $j \in N$? If E is reflexive, in particular if E is finite dimensional, this is easily seen to be true taking $E_j = E'$ if j is even and $E_j = E$ if E is odd. James, [6], constructs an example of a nonreflexive separable Banach space E isometric with its bidual. For such a space E the desired sequence $\{E_j\colon j \in N\}$ is readily produced. If E is required to be an infinite dimensional L-space the existence of such a sequence is impossible. This follows from the main result of this paper which asserts the existence of an integer, $n(E)$, for any infinite dimensional L-space E, such that E has no predual of order $n(E)$. If E is an L-space it is known that $E^{[2n]}$ is also an L-space for all n. Thus, for all n there are L-spaces E with $n(E)$ arbitrarily large. That $n(E)$ may be 0 was established by Gelfand, [4], for $E = L^1(\mu)$ where μ is Lebesgue measure on $[0, 1]$ or on $(-\infty, \infty)$, and by Pelcynski, [9], for μ σ-finite but not purely atomic. Rosenthal, [13], establishes that when Γ is a set whose cardinality, card(Γ), satisfies $\aleph_0 \leqslant \operatorname{card}(\Gamma) < 2^c$ then $l^\infty(\Gamma)$ does not have a predual of order 3 so that $l^1(\Gamma)$, although a dual space, is not a bidual space. Hence, $n(l^1(\Gamma)) = 1$. We shall remove the cardinality restriction Γ in Proposition 2. Thus, if μ is a σ-finite measure and $E = L^1(\mu)$ then $n(E) = 0$ unless μ is purely atomic in which case $n(E) = 1$.

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This appears to be the extent of precise knowledge of $n(E)$ although we shall give a method for obtaining crude estimates of $n(E)$ for more general L-spaces.

1. The preduals of order 2 of an L-space. In order to proceed we need to characterize Banach spaces whose nth duals are L-spaces. It is well known that the dual of an L-space (M-space) is an M-space with unit (L-space).

If F is an M-space which has a predual E then E considered as a subspace of F' is unique and is an L-space with the induced ordering from the L-space F'. If E is an L-space with predual F then F need not be an M-space with any ordering. Preduals of L-spaces E are called Lindenstrauss spaces. If a Lindenstrauss space, when given the induced ordering from E', is an ordered Banach space with E as order-dual it is called a simplex space. The classification of Lindenstrauss spaces is an unwieldy task which has absorbed the research of many persons. The situation for preduals of order $n > 1$ could be expected to be even more complicated. Somewhat surprisingly the situation is simplified enormously.

PROPOSITION 1. *Let E be a Banach space with E'' an L-space. E' is an M-space and E is an L-space.*

PROOF. Let B' and B'' be the unit balls of E' and E''. There is an extreme point e of B' since B' is $\sigma(E', E)$ compact. Lindenstrauss, [7], shows that there is a compact Hausdorff space X and an isometry of E' onto a separating subspace F of $\mathcal{C}(X)$ such that e is mapped to 1 and such that F has the Riesz separation property with respect to the ordering of $\mathcal{C}(X)$. With the ordering and norm from $\mathcal{C}(X)$ F is order unit normed with order unit e. Any such space is isometric and order isomorphic with the space $A(\Delta)$ of continuous affine functions on a Choquet simplex Δ. Thus E' is isometric with $A(\Delta)$. Capon shows, [2], that when $A(\Delta)$ is a dual space then Δ is a Bauer simplex. Thus, $A(\Delta)$ is an M-space as is E'. Since E is the predual of E' it is an L-space. □

COROLLARY 1-1. *Let E be a Banach space with no predual. If, for some integer n, $E^{[n]}$ is an L- or an M-space then E is either an L-space or a Lindenstrauss space.*

REMARKS. 1. The following simple category theoretic proof of Proposition 1 is due to K. S. Lau. For terminology and justification of all statements we refer the reader to [14]. We work in the category Ban_1 of Banach spaces with linear contractions as morphisms. Any dual M-space is injective and all injective objects in this category are M-spaces. If E'' is an L-space E' is a retract of the injective E''' hence is itself injective. Thus E' is an M-space and E is an L-space.

2. One might conjecture that if E were a Banach space with E'' linearly isomorphic to an L-space then E would also be linearly isomorphic to an L-space. A counterexample appears in Chapter 5d of [9]. An example is given

of a Banach space E not even linearly isomorphic to a complemented subspace of an L-space yet with E' linearly isomorphic with $l^\infty(N)$. Severe restrictions on E are necessary in order that E'' linearly isomorphic to an L-space imply the existence of an L-space linearly isomorphic with E.

PROPOSITION 2. *If Γ is an infinite set then $n(l^1(\Gamma)) = 1$. In fact $l^1(\Gamma)$ is not even linearly isomorphic to a double conjugate space.*

PROOF. Let Ban be the category of Banach spaces with continuous linear maps as morphisms. Let E be a Banach space with E'' isomorphic to $l^1(\Gamma)$. E''' is isomorphic to $l^\infty(\Gamma)$ which is injective in Ban_1 hence E''' is injective in Ban. Consequently, E' is injective in Ban. Corollary 3 of **[12]**, shows that E' contains a closed subspace l linearly isomorphic to $l^\infty(N)$. Since $l^\infty(N)$ is injective in Ban, l is complemented in E' hence E'' contains a complemented subspace linearly isomorphic to $l^{\infty\prime}(N)$ hence to $\mathfrak{M}(\beta N)$. Since βN is not scattered there is a nonatomic measure $\mu \in \mathfrak{M}^+(\beta N)$, **[13]**. If $\mathfrak{B}$ is a separable μ-nonatomic σ-algebra of subsets of N the space $L^1(\beta N, \mathfrak{B}, \mu)$ is isometric both with $L^1(\lambda_1)$ where λ_1 is Lebesgue measure on [0, 1] and with the closed subspace $\{f \cdot \mu\colon f \in L^1(\beta N, \mathfrak{B}, \mu)\} \subset \mathfrak{M}(\beta N)$. Thus, $l^1(\Gamma)$ contains a closed subspace linearly isomorphic to $L^1(\lambda_1)$. Since any separable subspace of $l^1(\Gamma)$ is in $l^1(\Lambda)$ where $\Lambda \subset \Gamma$ has cardinality $\aleph_0$, $l^1(N)$ contains a subspace linearly isomorphic with $L^1(\lambda_1)$ which is impossible by the remarks on p. 123 of **[9]**. □

PROPOSITION 3. *If E is a triple conjugate space with E'' linearly isomorphic to an L-space then E is linearly isomorphic to an L-space.*

PROOF. Let $G''' = E$. E'' is an $\mathfrak{L}_1$ space so by **[9**, II.5.8] G' and E are $\mathfrak{L}_1$ spaces whereas G, G'', and E' are $\mathfrak{L}_\infty$ spaces. From **[9**, II.5.7] it follows that G'' is injective in Ban. Haydon, **[5]** shows that a bidual space injective in Ban is linearly isomorphic to $l^\infty(\Gamma)$ for some Γ. Thus, E is linearly isomorphic to the L-space $l^{\infty *}(\Gamma)$.□

REMARKS. 3. If F is an $\mathfrak{L}_\infty$ space then F'' is a bidual space injective in Ban. Thus F'' and all higher even order duals of F are linearly isomorphic to M-spaces with the odd order duals linearly isomorphic to L-spaces. If E is an $\mathfrak{L}_1$ space then $E^{[2n]}$ is linearly isomorphic to an L-space and $E^{[2n-1]}$ is linearly isomorphic to an M-space if $n \geqslant 2$.

2. The main result. To establish our result we will show that the dimension of an infinite dimensional L-space is strictly less than that of its bidual. As in **[13]**, the dimension of an infinite dimensional Banach space F, $\dim(F)$, is the minimum cardinality of a total subset. Equivalently, $\dim(F)$ is the minimum cardinality of a dense subset, the minimum algebraic dimension of a dense subspace, or the maximum cardinality of a subset, T, distinct elements of which are at least a distance 2 apart.

To avoid cumbersome symbolism we shall denote, for cardinal numbers n

and m, n^m by either $\exp(n, m)$ or $\exp^1(n, m)$. If $k \in N$ we define, by induction, $\exp^{k+1}(n, m)$ to be $\exp(n, \exp^k(n, m))$.

If K is a compact Hausdorff space and $\mathfrak{M}(K) = \mathcal{C}'(K)$ then $\dim(\mathfrak{M}(K)) \geqslant \operatorname{card}(K)$ for the set $\delta(K) = \{\delta_x, x \in K\}$ is a subset such that $\|\delta_x - \delta_y\| = 2$ if $x \neq y$.

If m is a cardinal number we let $\bar{2}^m = \{0, 1\}^m$ and set λ^m equal to the fair coin toss measure on $\bar{2}^m$. If λ_m is the Lebesgue product measure on $[0, 1]^m$ it is known that $L^1(\lambda_m)$ is isometric with $L^1(\lambda^{m \cdot \aleph_0})$ hence with $L^1(\lambda^m)$ when m is infinite. When m is infinite $\dim(L^1(\lambda_m))$ is known to be m hence $\dim(L^1(\lambda^m)) = m$.

If $\{E_\alpha\}$ is a family of Banach spaces, the l^1-direct sum, $(\Sigma_\alpha E_\alpha)_1$, is the set of all $e = (e_\alpha) \in \Pi_\alpha E_\alpha$ such that $\|e\|_1 = \Sigma_\alpha \|e_\alpha\| < \infty$. The l^∞-direct sum, $(\Sigma_\alpha E_\alpha)_\infty$, is the set of all $e = (e_\alpha) \in \Pi_\alpha E_\alpha$ such that $\|e\|_\infty = \sup_\alpha \|e_\alpha\| < \infty$. $(\Sigma_\alpha E_\alpha)_1$, with the norm $\| \ \|_1$, is a Banach space whose dual is $(\Sigma_\alpha E'_\alpha)_\infty$ with the norm $\| \ \|_\infty$. The dimension of $(\Sigma_\alpha E_\alpha)_1$ is $\Sigma_\alpha \dim(E_\alpha)$ where $\dim(E_\alpha)$ is the algebraic dimension of E_α if E_α is finite dimensional, for, if T_a is total in E_α for all α then $\cup_\alpha T_\alpha$ is total in $(\Sigma_\alpha E_\alpha)_1$. If Γ is a set with cardinality m then $\dim(l^1(\Gamma)) = m$ for $l^1(\Gamma)$ is an l^1-direct sum of one dimensional spaces.

If G is a Banach space with $m = \dim(F) \geqslant \aleph_0$ and T is a total subset of G then G' maybe considered to be a subset of $(-\infty, \infty)^T$. Since

$$\operatorname{card}\big((-\infty, \infty)^T\big) = \exp\big(\exp(2, \aleph_0), m\big) = \exp(2, m)$$

when $m \geqslant \aleph_0$ $\dim(G') \leqslant \operatorname{card}(G') \leqslant \exp(2, m)$. Similarly, $\dim(G'') \leqslant \exp^2(2, m)$ and, in general, $\dim(G^{[k]}) \leqslant \exp^k(2, m)$ for all $k \in N$.

If $m \geqslant \aleph_0$ is the cardinal of a set Γ then $l^\infty(\Gamma)$ is isometric with $\mathcal{C}(\beta\Gamma)$ hence

$$\dim\big(l^{\infty'}(\Gamma)\big) = \dim(\mathfrak{M}(\beta\Gamma)) \geqslant \operatorname{card}(\beta\Gamma) = \exp^2(2, m).$$

Since $l^{\infty'}(\Gamma) = [l^1(\Gamma)]''$, $\dim(l^{\infty'}(\Gamma)) = \exp^2(2, m)$.

If $m \geqslant \aleph_0$ then $\mathcal{C}(\bar{2}^m)$ is isometric to a subspace of $L^\infty(\lambda^m)$ hence $\mathfrak{M}(\bar{2}^m)$ is isometric to a quotient space of $L^{\infty'}(\lambda^m)$. Thus $\dim(L^{\infty'}(\lambda^m)) \geqslant \dim(\mathfrak{M}(\bar{2}^m)) \geqslant \operatorname{card}(\bar{2}^m) = 2^m$.

PROPOSITION 4. *If E is an L-space with* $m = \dim(E) \geqslant \aleph_0$ *then* $\exp(2, m) \leqslant \dim(E'') \leqslant \exp^2(2, m)$.

PROOF. It is only necessary to show that $\exp(2, m) \leqslant \dim(E'')$.

We may assume by the Kakutani-Maharam Representation Theorem, **[14]**, that $E = [l^1(\Gamma) + \Sigma_{\alpha \in \Lambda} L^1(\lambda^{m_\alpha})]_1$ with Γ and Λ sets and $m_\alpha \geqslant \aleph_0$ for $\alpha \in \Lambda$.

Case 1. $m = \operatorname{card}(\Gamma)$. E contains $l^1(\Gamma)$ isometrically hence E'' contains $l^{\infty'}(\Gamma)$ isometrically. Thus

$$\dim(E'') \geqslant \dim\big(l^{\infty'}(\Gamma)\big) = \exp^2(2, m) > \exp(2, m).$$

Case 2. $m = m_\alpha$ for some $\alpha \in \Lambda$. E contains $L^1(\lambda^m)$ isometrically hence

$$\dim(E'') \geqslant \dim\big(L^{\infty'}(\lambda^m)\big) \geqslant \exp(2, m).$$

Case 3. $m > \operatorname{card}(\Gamma)$ and $m > m_\alpha$ if $\alpha \in \Lambda$. In this case

$$m = \sum_{\alpha \in \Lambda} m_\alpha = \sup_{\alpha \in \Lambda} m_\alpha.$$

Regard each m_α as the first ordinal of cardinal m_α. Well order Λ so that $\{m_\alpha : \alpha \in \Lambda\}$ is nondecreasing along Λ. If the ordinals m and m_α are regarded as sets then m is the increasing union $\bigcup_{\alpha \in \Lambda} m_\alpha$. Let $\pi_\alpha : \bar{2}^m \to \bar{2}^{m_\alpha}$ be the natural projection restricting a function on the set m to the smaller set m_α. Let $\mathcal{F}_\alpha$ be the Borel sets in $\bar{2}^{m_\alpha}$, let $\mathcal{F}$ be the Borel sets in $\bar{2}^m$ and let $\mathcal{G}_\alpha = \pi_\alpha^{-1}(\mathcal{F}_\alpha)$ for $\alpha \in \Lambda$. The set $\{\mathcal{G}_\alpha \alpha \in \Lambda\}$ is an increasing family of σ-algebras. The inverse image of the measure λ^{m_α} on $\mathcal{G}_\alpha$ under π_α is the restriction of λ^m to $\mathcal{G}_\alpha$. Equivalently, λ^{m_α} is the image of λ^m on $\bar{2}^{m_\alpha}$ under π_α. The map j_α: $f \to f \circ \pi_\alpha$ yields an isometric embedding of $L^1(\lambda^{m_\alpha})$ into $L^1(\lambda^m)$ with range $L^1(\bar{2}^m, \mathcal{G}_\alpha, \lambda^m)$. If $f = (f_\alpha) \in [\Sigma_{\alpha \in \Lambda} L^1(\lambda^{m_\alpha})]_1$ define $j(f)$ as $\Sigma_{\alpha \in \Lambda} j_\alpha(f_\alpha)$. The map j is of norm 1 from $[\Sigma_{\alpha \in \Lambda} L^1(\lambda^{m_\alpha})]_1$ into $L^1(\lambda^m)$. If $g \in L^1(\lambda^m)$ and $\alpha \in \Lambda$ then $E(g|\mathcal{G}_\alpha)$ is in the range of j_α hence is in that of j and $\|E(g|\mathcal{G}_\alpha)\| \leqslant \|g\|$. By the Martingale Convergence Theorem,

$$\lim_{\alpha \in \Lambda} E(g|\mathcal{G}_\alpha) = g$$

in $L^1(\lambda^m)$. This shows that the image of the unit ball in $[\Sigma_{\alpha \in \Lambda} L^1(\lambda^{m_\alpha})]_1$ is dense in that of $L^1(\lambda^m)$. From this it follows that j is a surjection and that $L^1(\lambda^m)$ is isometric to the quotient of $\Sigma_{\alpha \in \Lambda} L^1(\lambda^{m_\alpha})$ by $\ker(j)$ hence that $L^1(\lambda^m)$ is isometric to a quotient space of E. Thus, $L^{\infty\prime}(\lambda^m)$ is isometric with a quotient space of E'' and $\dim(E'') \geqslant \dim(L^{\infty\prime}(\lambda^m)) \geqslant \exp(2, m)$. $\square$

COROLLARY 4-1. *Let E be an L-space with $m = \dim(E) \geqslant \aleph_0$. If $k \in N$ then* $\exp^k(2, m) \leqslant \dim(E^{[2k]}) \leqslant \exp^{2k}(2, m)$.

COROLLARY 4-2. *If E is an L-space with $\aleph_0 \leqslant \dim(E) \leqslant \exp^k(2, \aleph_0)$ for some $k \in N$ then $n(E) < 2k$.*

COROLLARY 4-3. *If E is an L-space with* $\dim(E) \geqslant \exp^k(2, \aleph_0)$ *for all $k \in N$ then E has no separable predual of any order.*

COROLLARY 4-4. *If F is an infinite dimensional Banach space and $k \in N$ with $F^{[k]}$ linearly isomorphic to an L-space then $F^{[n]}$ and $F^{[m]}$ are not linearly isomorphic for any $n \neq m$ in N.*

This last corollary is very well known.

COROLLARY 4-5. *If E is an infinite dimensional L-space it is nonreflexive.*

PROPOSITION 5. *If E is an infinite dimensional $\mathcal{L}_1$ (or $\mathcal{L}_\infty$ space) there is an integer $\bar{n}(E) < \infty$ so that if F is linearly isomorphic to E then $n(F) \leqslant \bar{n}(E)$.*

PROOF. Let C be the set of cardinal numbers m so that there is an L-space F with E linearly isomorphic to $F^{[2n]}$ for some integer n and with $m = \dim(F)$. Let $M = \min(C)$. Let k_0 be an integer and let F_0 be an L-space with $\dim(F_0) = M$ and with $F_0^{[2k_0]}$ linearly isomorphic to E. If G is any L-space

and j is an integer with $G^{[2j]}$ linearly isomorphic with E, Corollary 4-1 and the fact that $\dim(G) \geqslant M$ implies that

$$\exp^j(2, M) \leqslant \dim(G^{[2j]}) = \dim(E) = \dim\left(F_0^{[2k_0]}\right) \leqslant \exp^{2k_0}(2, M).$$

Thus $j \leqslant 2k_0$.

If E is not isomorphic with a fourth conjugate space then $\bar{n}(E) \leqslant 3$. Otherwise, if E is linearly isomorphic to $F^{[2k]}$ with $k \geqslant 2$ Proposition 3 and Remarks 3 show that $F^{[4]}$ is isomorphic to an L-space G. Thus $k - 2 \leqslant 2k_0$. Allowing for the possibility that F is a dual space it follows that $\bar{n}(E) < 2k_0 + 3$. □

COROLLARY 5-1. *If E is an L-space (or M-space) then $n(E) < \infty$.*

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