LOGCONCAVITY OF THE COOLING OF A CONVEX BODY

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Abstract. The solution $u(x, t)$ of the heat equation is logconcave in the space variable $x$ whenever the initial temperature $u_0(x)$ of the convex body is logconcave.

1. Introduction. A real function $f$ defined over a convex set is called logconcave if $f$ is a nonnegative function on $\Omega$ and if the following inequalities hold:

$$f((1-t)x + ty) \geq (f(x))^{1-t} (f(y))^t$$

when $x$ and $y$ belong to $\Omega$ and when $t$ is between 0 and 1. Let us consider a solution to the heat equation: $u(x, t)$ is defined for $x$ belonging to a bounded convex open subset of $\mathbb{R}^d$ and for $t > 0$ and we assume that

$$\frac{\partial u}{\partial t} = \frac{1}{2} \Delta u.$$ 

The following boundary conditions will be used:

(a) For every boundary point $a$ of $\Omega$ and every $t > 0$, $\lim_{x \to a} u(x, t) = 0$.

(b) For every point $x$ of $\Omega$, $\lim_{t \to 0} u(x, t) = u_0(x)$ where $u_0(x)$ is a given function defined on $\Omega$.

The main result we will show is that the function $u(x, t)$ is logconcave in the space variable $x$ whenever the function $u_0(x)$ is logconcave. Otherwise said, if the initial temperature of a convex body is logconcave, at any time later the temperature of the body will still be logconcave in the space variable. From this fact, we will derive a property of Helmholtz's equation. If $\Omega$ is a bounded convex open subset of $\mathbb{R}^d$ and if $v(x)$ is a nonnegative eigenfunction for the Laplace equation: $\Delta v(x) = \lambda v(x), x \in \Omega$, with boundary conditions $\lim_{x \to a} v(x) = 0, a \in \partial \Omega$, then $v(x)$ is a logconcave function.

The basic tool for our study comes from Prékopa, it says that the convolution of two logconcave kernels is a logconcave kernel.

Theorem 1 (Prékopa [4]). Let be $f(x, y)$ and $g(y, z)$ two logconcave functions, the first defined on $\mathbb{R}^p \times \mathbb{R}^q$, the second defined on $\mathbb{R}^q \times \mathbb{R}^r$, then the function $h(x, z) = \int_{\mathbb{R}^q} f(x, y) g(y, z) \, dy$ is logconcave over the set where it is finite.

Received by the editors July 17, 1978.


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0002-9939/79/0000-0216/$02.00
If Prékopa's proof is a little complicated, Brascamp and Lieb [1] found a simpler proof which uses the Brunn-Minkowsky-Lusternik inequality.

2. Logconcavity of the kernel of the heat equation. If $\Omega$ is an open subset of $\mathbb{R}^d$, one way to solve the heat equation on the cylinder $\Omega \times (0, \infty)$ is to introduce a kernel $p(x, y; t) = p(x, y; t)$. If $\Delta_x$ is the Laplace operator with respect to $x$-variable, one looks for a function $p$ such that

$$
\frac{\partial p}{\partial t}(x, y; t) = \frac{1}{2} \Delta_x p(x, y; t), \quad x \in \Omega, y \in \Omega, t > 0,
$$

$$
\lim_{x \to a} p(x, y; t) = 0, \quad a \in \partial \Omega, y \in \Omega, t > 0,
$$

$$
\lim_{t \to 0} \int_{\Omega} p(x, y; t)u_0(y)dy = u_0(x), \quad x \in \Omega.
$$

The last equation should hold for any bounded continuous function $u_0$ on $\Omega$. The previous equation should hold for any regular boundary point $a$ of $\partial \Omega$.

The construction of such a kernel is well described by Itô and McKean [2, pp. 238–239]. We recall their explanations. We denote by $\|y\|$, the Euclidean norm in $\mathbb{R}^d$ of a vector $y$ of $\mathbb{R}^d$. If

$$
f_n(x_0, x_1, \ldots, x_n; t) = (2\pi t/n)^{-d/2} \exp \left\{ \left( \sum_{i=1}^{n-1} ||x_i - x_0||^2 + ||x_2 - x_1||^2 + \cdots + ||x_n - x_{n-1}||^2 \right) n/(2t) \right\}
$$

where $t > 0$, $x_i \in \mathbb{R}^d$, $0 < i < n$, one defines $p_n(x_0, x_n; t)$ as the integral in $\mathbb{R}^{d(n-1)}$ of the function $f_n$ over the set $\{(x_1, x_2, \ldots, x_{n-1}) : x_i \in \Omega, 1 < i < n - 1\}$. It is convenient to restrict $n$ to powers of 2. It is easy to check that $p_{2^k}(x, y; t)$, $k = 0, 1, 2, \ldots$, is a decreasing sequence of functions. The limit $p(x, y; t) = \lim_{k \to \infty} p_{2^k}(x, y; t)$ is the kernel of the heat equation. Something more can be said about this kernel when $\Omega$ is a convex open subset of $\mathbb{R}^d$.

Theorem 2. If $\Omega$ is a convex open subset of $\mathbb{R}^d$, the kernel of the heat equation $p(x, y; t)$ is logconcave with respect to $x$ and $y$.

Proof. If we define the function

$$
g_n(x_0, x_1, \ldots, x_n; t) = \begin{cases} f_n(x_0, x_1, \ldots, x_n; t) & \text{if } x_i \in \Omega, 0 < i < n, \\ 0 & \text{if } (\exists i) x_i \notin \Omega; \end{cases}
$$

$g_n(x_0, x_1, \ldots, x_n; t)$ is a logconcave function in the variables $x_0, x_1, \ldots, x_n$. By Prékopa’s theorem, integration with respect to $x_1, x_2, \ldots, x_{n-1}$ will leave this property for the result of integration, $p_n(x_0, x_n; t)$, $p(x, y; t)$ as a limit of logconcave functions will still be logconcave.

3. Logconcavity and the cooling of a convex body. We say that a measure $\mu$ on $\mathbb{R}^d$ is logconcave if the support of the measure is a convex subset $K$ of $\mathbb{R}^d$ and if there is a logconcave function $f(x)$ defined on $K$ such that $d\mu(x) = f(x)dH(x)$ where $H$ is the linear manifold generated by $K$ and $dH(x)$ is the Lebesgue measure on $H$. 
Theorem 3. If $\Omega$ is a convex open subset of $\mathbb{R}^d$, and if $\mu_0$ the initial distribution of temperature of $\Omega$ is a logconcave measure carried by $\Omega$, then the distribution of temperature $u(x, t)$ at time $t$ is a logconcave function of $x$:

$$u(x, t) = \int_{\Omega} p(x, y; t) \, d\mu_0(y).$$

Proof. Theorems 1 and 2 give the proof.

4. Helmholtz's equation and logconcavity. We now restrict our attention to a bounded open convex subset $\Omega$ of $\mathbb{R}^d$. Helmholtz's equation is $\Delta v = \lambda v$ where $v \in C^2(\Omega), \lambda \in \mathbb{R}$. It is known that $H_\lambda = \{ v \in C^2(\Omega) \cap C(\overline{\Omega}) : \Delta v = \lambda v, \quad (\forall x \in \partial \Omega) v(x) = 0 \}$ is a finite dimensional vector space and that $L^2(\Omega)$ is the orthogonal sum of all these $H_\lambda$. Moreover $H_\lambda = \{0\}$ if $\lambda > 0$ and $\{\lambda : H_\lambda \neq \{0\}\}$ is a countable set with no accumulation point in any finite interval. Let us denote by $\lambda_1, \lambda_2, \ldots$ the decreasing sequence of these $\lambda$ such that $H_{\lambda_n} \neq \{0\}$. If $r_n$ is the dimension of $H_{\lambda_n}$ and if one chooses an orthonormal basis $v_{n,1}, v_{n,2}, \ldots, v_{n,r_n}$ of $H_{\lambda_n}$, the family of functions $\{v_{n,i} : 1 < i < r_n, \quad n = 1, 2, \ldots\}$ is an orthonormal basis of $L^2(\Omega)$. The connection between this basis and the kernel of the heat equation is the spectral representation of the kernel

$$p(x, y; t) = \sum_{n=1}^{\infty} \sum_{i=1}^{r_n} e^{\lambda_i t} v_{n,i}(x)v_{n,i}(y).$$

These results can be found in Kac [3] for example.

Up to this point, the convexity of $\Omega$ is not needed. When $\Omega$ is convex (or more generally if $\Omega$ is connected), something more is known about the spectral value $\lambda_1$: $r_1 = 1$ and $v_1(x) = v_{1,1}(x)$ does not vanish on $\Omega$. One can always assume that $v_1(x) > 0$; if this is not the case, interchange $v_1$ with $-v_1$. This unique function $v_1$ will be called the dominant eigenfunction of Laplace operator.

Theorem 4. If $\Omega$ is a bounded convex open subset of $\mathbb{R}^d$, the dominant (positive) eigenfunction of Laplace operator is logconcave.

Proof. To prove that $v_1$ is logconcave, it suffices to prove that $p(x, y; t)e^{-\lambda_1 t}$ converges to $v_1(x)v_1(y)$ as $t$ tends to $\infty$, since the heat kernel is logconcave. By using the spectral representation of $p(x, y; t)$, we get

$$p(x, y; t)e^{-\lambda_1 t} - v_1(x)v_1(y) = \sum_{n=2}^{\infty} \sum_{i=1}^{r_n} e^{\lambda_i t - \lambda_1 t} \left( \sum_{i=1}^{r_n} v_{n,i}(x)v_{n,i}(y) \right).$$

By the Cauchy-Schwarz inequality,

$$|p(x, y; t)e^{-\lambda_1 t} - v_1(x)v_1(y)| \leq \left( \sum_{n=2}^{\infty} \sum_{i=1}^{r_n} e^{\lambda_i t - \lambda_1 t} v_{n,i}^2(x) \right)^{1/2} \left( \sum_{n=2}^{\infty} \sum_{i=1}^{r_n} e^{\lambda_i t - \lambda_1 t} v_{n,i}^2(y) \right)^{1/2}.$$

In order to get bounds on each of these square roots, we use again the
spectral representation with $y = x$ and $t = 1$: 

$$\sum_{n=1}^{\infty} \sum_{i=1}^{r_n} e^{\lambda_n v_n^2(x)} = p(x, x; 1) < (2\pi)^{-d/2}.$$ 

So, if $M(t) = \sup_{n \geq 2} e^{\lambda_n - \lambda_n t - \lambda_n}$, then 

$$|p(x, y; t) e^{-\lambda_n t} - v_1(x) v_1(y)|$$ 

$$\leq M(t) \left( \sum_{n=2}^{\infty} \sum_{i=1}^{r_n} e^{\lambda_n v_n^2(x)} \right)^{1/2} \left( \sum_{n=2}^{\infty} \sum_{i=1}^{r_n} e^{\lambda_n v_n^2(y)} \right)^{1/2}$$ 

$$\leq M(t)(2\pi)^{-d/2}.$$ 

Since $\lim_{t \to \infty} M(t) = 0$, the proof is completed.

REFERENCES


