PROCEEDINGS OF THE
AMERICAN MATHEMATICAL SOCIETY
Volume 74, Number 2, May 1979

A CONSTRUCTION OF HILBERT SPACES OF ANALYTIC FUNCTIONS[1]

WILLIAM W. HASTINGS

ABSTRACT. A simple technique is presented for constructing a measure ν from a given measure so that $R^2(K, \nu)$ has certain properties. Here, $R^2(K, \nu)$ is the closure in $L^2(\nu)$ of the rational functions with poles off K, a compact set containing the support of ν. A typical example shows that there exists a measure ν mutually absolutely continuous with area measure on the unit disk such that $\sqrt{z}$ (principal branch) is an element of $R^2(K, \nu)$, while each point of the open disk except on the negative real axis is an analytic bounded point evaluation for $R^2(K, \nu)$.

The purpose of this paper is to construct various examples of Hilbert spaces of analytic functions. A typical example involves the construction of a measure ν which is mutually absolutely continuous with respect to area measure on the unit disk and for which each element of $H^2(\nu)$ (see definition below) is analytic in the unit disk except possibly on a fixed line segment. Other examples show that $H^2(\nu)$ can "split" (into a direct sum) in a natural way. These examples have implications in the study of subnormal operators and bounded point evaluations.

If E is a compact subset of the complex plane $\mathbf{C}$, let $C(E)$ be the space of continuous functions on E with the supremum norm. If K is compact and $K \supset E$, let $R(K, E)$ be the closure in $C(E)$ of the rational functions with poles off K. Let μ be a measure with compact support spt μ. (By "measure" we will always mean a finite, positive Borel measure on the complex plane with compact support.) If K is a compact set containing spt μ, let $R^2(K, \mu)$ be the closure in $L^2(\mu)$ of $R(K, \operatorname{spt} \mu)$. If K is the polynomially convex hull of spt μ, then $R^2(K, \mu) = H^2(\mu)$, the closure in $L^2(\mu)$ of the analytic polynomials. If μ and ν are mutually absolutely continuous measures, we will write $\mu \sim \nu$.

A point $\lambda \in \mathbf{C}$ is said to be a bounded point evaluation (b.p.e.) for $R^2(K, \mu)$ if there exists a constant $C > 0$ such that $|r(\lambda)| \leq C\|r\|_\mu$ for every rational function r with poles off K. (Here, $\|r\|_\mu$ is the $L^2(\mu)$-norm of r.) If λ is a b.p.e. for $R^2(K, \mu)$, then $f(\lambda)$ is well defined for each $f \in R^2(K, \mu)$. Let $\Lambda(K, \mu)$ be the set of all b.p.e.'s for $R^2(K, \mu)$. If $\lambda \in \operatorname{int} \Lambda(K, \mu)$, the interior of $\Lambda(K, \mu)$, then λ is said to be an analytic b.p.e. if the mapping $z \mapsto f(z)$ is

---

Presented to the Society, October 29, 1977; received by the editors September 18, 1978.
AMS (MOS) subject classifications (1970). Primary 46E20; Secondary 47B20.
[1]Research for this paper was supported in part by an NSF Research Grant No. MSC77-01811.

0002-9939/79/0000-0217/$02.00

analytic at λ for every $f \in R^2(K, \mu)$. Let $\Lambda_a(K, \mu)$ be the set of all analytic b.p.e.'s for $R^2(K, \mu)$.

LEMMA. *Suppose f is a μ-measurable function for which there exist compact sets $E_1 \subset E_2 \subset \cdots \subset \operatorname{spt} \mu$ and functions $g_n \in R(K, E_n)$ such that $\lim_{n\to\infty} \mu(E_n) = \mu(\mathbf{C})$ and $f(z) = g_n(z)$ for almost every (with respect to μ) $z \in E_n$. Then there exist a measure ν and constants $c_n > 0$ such that*

(a) $\nu \sim \mu$,

(b) $\nu|_{E_n} \geqslant c_n \mu|_{E_n}$,

(c) $f \in R^2(K, \nu)$.

PROOF. The technique of this proof is due to J. Bram [1, Theorem 6]. For each $n \geqslant 1$, choose a rational function r_n with poles off K such that $|f(z) - r_n(z)| < 1/n$ for almost every $z \in E_n$. Let $M_n = \max\{|r_n(z)|\colon z \in \operatorname{spt} \mu\}$. Define ν as follows: $\nu|_{E_1} = \mu|_{E_1}$ and $\nu|_{E_{n+1}\setminus E_n} = c_n \mu|_{E_{n+1}\setminus E_n}$, where $\{c_n\}_{n\geqslant 1}$ is chosen with $1 \geqslant c_1 \geqslant c_2 \geqslant c_3 \geqslant \cdots > 0$ such that $2M_n^2 c_n \leqslant n^{-2}$ and

$$\sum_{n=1}^{\infty} c_n \int_{E_{n+1}\setminus E_n} |f|^2 \, d\mu < \infty.$$

The last condition is possible because by hypothesis f is essentially bounded on $E_{n+1} \setminus E_n$. With these conditions on c_n, (a) and (b) are clearly satisfied, and (where A' denotes the complement of A in $\mathbf{C}$)

$$\begin{aligned}
\|f - r_n\|_\nu^2 &= \int_{E_n} |f - r_n|^2 \, d\nu + \int_{E_n'} |f - r_n|^2 \, d\nu \\
&\leqslant n^{-2}\mu(E_n) + 2c_n \int_{E_n'} |r_n|^2 \, d\mu + 2\int_{E_n'} |f|^2 \, d\nu \\
&\leqslant n^{-2}\mu(\mathbf{C}) + 2 \sum_{k=n}^{\infty} c_k \int_{E_{k+1}\setminus E_k} |f|^2 \, d\mu.
\end{aligned}$$

Hence $r_n \to f$ in $R^2(K, \nu)$.

EXAMPLE 1. Let $\mathbf{D}$ be the open unit disk and let $\mu = \mu_1 + \mu_2$, where μ_1 is Lebesgue area measure restricted to $\mathbf{D}$ and μ_2 is Lebesgue linear measure on $\partial \mathbf{D}$. Then there exists a measure $\nu \sim \mu$ such that the characteristic function of $\partial \mathbf{D}$ belongs to $H^2(\nu)$ and $\Lambda_a(\nu) = \Lambda_a(\overline{\mathbf{D}}, \nu) = \mathbf{D}$. Thus $H^2(\nu)$ "splits" naturally into

$$H^2(\nu) = H^2(\nu|_{\mathbf{D}}) \oplus H^2(\nu|_{\partial\mathbf{D}})$$

and each element of $H^2(\nu|_{\mathbf{D}})$ is analytic on $\mathbf{D}$. This splitting phenomenon is studied further by Kriete [5]. It is easily seen (cf. [2]) that $H^2(\nu|_{\partial\mathbf{D}}) = L^2(\nu|_{\partial\mathbf{D}})$.

PROOF. Let f be the characteristic function of $\partial \mathbf{D}$ and let

$$E_n = \{z\colon |z| \leqslant 1 - 1/n\} \cup \{z \in \partial\mathbf{D}\colon |\arg z| \geqslant 1/n\},$$

where $-\pi < \arg z \leqslant \pi$. By Runge's theorem, $f \in R(\overline{\mathbf{D}}, E_n)$ for each $n \geqslant 2$. Hence the result follows from the lemma.

PROPOSITION. *Let μ be a measure and K a compact set containing the support of μ. Suppose $U \subset K$ is an open subset of* $\mathbf{C}$ *such that $\mu(K \setminus U) = 0$ and U' has a finite number of components each of which intersects K'. For any μ-measurable function f analytic on U, there exist a measure ν and constants $c_n > 0$ satisfying* (a), (b) *and* (c) *of the lemma.*

PROOF. Let $E_n = \{z \in U\colon \operatorname{dist}(z, U') \geqslant 1/n\}$. We claim that each component of E_n' contains a component of U'. Indeed, suppose O is a component of E_n'. If $O \cap U' = \varnothing$, then O is an open subset of U, which is easily seen to contradict the definition of E_n. Hence there exists $z \in O \cap U'$. Since O is open and $\partial O \subset U$, the component of U' containing z is a subset of O and the claim is established. It follows that E_n' has only a finite number of components and each component of E_n' intersects K'. Therefore, $f|_{E_n} \in R(E_n, E_n)$ (cf. [**4**, Theorem II.10.4]). Furthermore, $R(E_n, E_n) = R(K, E_n)$, because the set of points z_0 for which the function $(z - z_0)^{-1}$ belongs to $R(K, E_n)$ is an open and relatively closed subset of E_n' containing at least one point from each component of E_n'. The result now follows from the lemma.

COROLLARY. *Suppose μ is Lebesgue area measure restricted to an open set G with $G \subset K$. Suppose U is an open subset of G such that $\mu(U') = 0$ and U' has a finite number of components each of which intersects K'. If f is analytic on U, then there exists a measure $\nu \sim \mu$ such that $f \in R^2(K, \nu)$ and $U \subset \Lambda_a(K, \nu)$.*

PROOF. Construct ν as indicated in the proposition and the lemma. Since ν is greater than a constant multiple of area measure on E_n, each point of int E_n is an analytic b.p.e. for $R^2(K, \nu)$. Since $U = \bigcup_n \operatorname{int} E_n$, the corollary follows.

REMARK. Let S be "multiplication by z" on $R^2(K, \nu)$, i.e., $(Sf)(z) = zf(z)$. Then S^* belongs to a class of operators $\mathfrak{B}_1(U)$ defined by Cowen and Douglas [**3**].

EXAMPLE 2. Suppose μ is Lebesgue area measure restricted to $\mathbf{D}$. Let $U = \mathbf{D} \setminus (-1, 0]$ and let $f(z) = \sqrt{z}$ or $f(z) = \log z$ (principal branch in either case). Then by the corollary, there exists a measure $\nu \sim \mu$ for which $f \in H^2(\nu)$ and U is the set of analytic b.p.e.'s for $H^2(\nu)$. Since f maps U onto a convex set, it is clear that $\Lambda(\nu) = U$.

EXAMPLE 3. The same as Example 2 except $f(z) = 1/z$ or $f(z) = (2z^2 + z)^{1/2}$. In either case, $\Lambda_a(\nu) = U$. The referee has pointed out that one can show, with a little work, that no point of the interval $(-1, 0]$ can be a b.p.e. Thus, as above, $\Lambda(\nu) = U$.

EXAMPLE 4. Take $U = \{z \in \mathbf{D}\colon \operatorname{Im} z \neq 0\}$ and let f be the characteristic function of the upper half-plane. Then f is analytic on U. Hence there exists a measure $\nu \sim \mu$ for which $\Lambda_a(\nu) = \Lambda(\nu) = U$ and for which $H^2(\nu)$ "splits" naturally into $H^2(\nu) = H^2(\nu_1) \oplus H^2(\nu_2)$, where ν_1 is ν restricted to the upper half-plane and $\nu_2 = \nu - \nu_1$.

References

1. J. Bram, *Subnormal operators*, Duke Math. J. **22** (1955), 75–94.

2. W. S. Clary, *Quasi-similarity and subnormal operators*, Dissertation, Univ. of Michigan, 1973.

3. M. J. Cowen and R. G. Douglas, *Complex geometry and operator theory*, Bull. Amer. Math. Soc. **83** (1977), 131–133.

4. T. W. Gamelin, *Uniform algebras*, Prentice-Hall, Englewood Cliffs, N. J., 1969.

5. Thomas Kriete, *The growth of point evaluation functionals in certain $H^2(\mu)$ spaces* (preprint).

Department of Mathematics, Fordham University, Bronx, New York 10458