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DUAL A^*-ALGEBRAS OF THE FIRST KIND

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ABSTRACT. Let A be an A^*-algebra of the first kind. It is proved that A has property P2 of Máté if and only if A^2 is dense in A if and only if A possesses an (operator-bounded) approximate identity. Further, it is shown that an A^*-algebra of the first kind having property P2 is a dual algebra if and only if it is a modular annihilator algebra. As applications, these results are used to strengthen certain theorems about Hilbert algebras.

1. Let $A = (A, \|\cdot\|)$ be an A^*-algebra, and let $\mathfrak{A} = (\mathfrak{A}, |\cdot|)$ be its C^*-algebra completion with respect to the auxiliary norm $|\cdot|$ [**11**, p. 181]. If A is a (dense two-sided) ideal of $\mathfrak{A}$, then A is called an A^*-algebra of the first kind. In this case, there is a constant $K > 0$ such that

$$\|wa\| \leqslant K|w|\,\|a\|, \quad \|aw\| \leqslant K|w|\,\|a\|,$$

for all a in A, w in $\mathfrak{A}$ [**1**, Proposition 2.2, Theorem 2.3]; thus, A is a Banach $\mathfrak{A}$-bimodule. In this paper, several characterizations of Máté's property P2 [**9**] are given for A^*-algebras of the first kind, and it is shown that such an algebra is dual if and only if it is modular annihilator. These results are used to strengthen some theorems in [**16**] concerning Hilbert algebras.

2. Let A be a Banach algebra with Banach space dual A^*, and let $A \hat{\otimes} A^*$ denote the projective tensor product of A and A^* [**12**, p. 93]. For a in A and f in A^*, define $f * a$ in A^* by $\langle f * a, b\rangle = \langle f, ab\rangle$, all b in A. It is immediate that A^* is a right Banach A-module with respect to this product. Moreover, by the universal property of the projective tensor product, there is a unique continuous linear map B: $A \hat{\otimes} A^* \to A^*$ such that $B(a \otimes f) = f * a$. For a general tensor

$$\zeta = \sum_{k=1}^{\infty} a_k \otimes f_k$$

in $A \hat{\otimes} A^*$, where $a_k \in A$, $f_k \in A^*$, and $\sum_{k=1}^{\infty}\|a_k\|\,\|f_k\| < +\infty$, it follows that $B(\zeta) = \sum_{k=1}^{\infty} f_k * a_k$. It also follows from the universal property of projective tensor product that there is a unique continuous linear map E: $A \hat{\otimes} A^* \to \mathbf{C}$ such that $E(a \otimes f) = \langle f, a\rangle$; more generally, for a tensor ζ in $A \hat{\otimes} A^*$ as above $E(\zeta) = \sum_{k=1}^{\infty}\langle f_k, a_k\rangle$. The Banach algebra A is said to have

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property P2 if $\ker(B) \subseteq \ker(E)$; in other words, if

$$\sum_{k=1}^{\infty} \|a_k\| \, \|f_k\| < +\infty \quad \text{and} \quad \sum_{k=1}^{\infty} f_k * a_k = 0,$$

then $\sum_{k=1}^{\infty}\langle f_k, a_k\rangle = 0$ **[9]**, [**14**, Theorem 2.3].

For a Banach algebra A to possess property P2, A must have a right approximate identity [**7**]. In this paper, every approximate identity $\{u_\lambda\}$ will be two-sided (i.e., $\|u_\lambda a - a\| \to 0$, $\|au_\lambda - a\| \to 0$, for all a in A). This is no restriction since the algebras under consideration all have continuous involution. An approximate identity $\{u_\lambda\}$ for A is said to be operator-bounded if

$$\sup\{\|u_\lambda a\|, \|au_\lambda\| : a \in A, \|a\| \leqslant 1\} < +\infty.$$

Finally, A^2 will denote the linear span of products ab, where $a, b \in A$.

THEOREM 1. *If A is an A^*-algebra of the first kind, then the following statements are equivalent:*

(a) *A^2 is dense in A.*

(b) *A has property* P2.

(c) *A has an approximate identity.*

(d) *A has an operator-bounded approximate identity.*

PROOF. The implications (d) $\Rightarrow$ (c) $\Rightarrow$ (a) are immediate; therefore, it suffices to show that (b) $\Rightarrow$ (a) $\Rightarrow$ (d) $\Rightarrow$ (b).

((b) $\Rightarrow$ (a)). Suppose that A has property P2. Then, by [**14**, Theorem 2.3], there exists a net $\{u_\lambda\}$ in A such that $\langle f, au_\lambda\rangle \to \langle f, a\rangle$, for all a in A, f in A^*. It follows that no nonzero element f of A^* can annihilate A^2; thus, (a) holds.

((a) $\Rightarrow$ (d)). Let $\mathfrak{A} = (\mathfrak{A}, |\cdot|)$ be the C^*-algebra completion of $A = (A, \|\cdot\|)$ in the auxiliary norm $|\cdot|$. Because A is a dense ideal of $\mathfrak{A}$, it contains a (two-sided) approximate identity $\{u_\lambda\}$ for $\mathfrak{A}$ such that $u_\lambda^* = u_\lambda$ and $\|u_\lambda\| \leqslant 1$, for all λ [**3**, Proposition 1.7.2]. Now, since A is a Banach $\mathfrak{A}$-bimodule, by the Hewitt-Cohen factorization theorem [**4**, Theorem 32.22], $\mathfrak{A} \cdot A = \{wa : w \in \mathfrak{A}, a \in A\}$ (resp., $A \cdot \mathfrak{A}$) is a closed linear subspace of A. However, $A^2 \subseteq \mathfrak{A} \cdot A \cap A \cdot \mathfrak{A}$, and A^2 is dense in A by hypothesis. Therefore, $A = \mathfrak{A} \cdot A = A \cdot \mathfrak{A}$, and it follows easily that $\{u_\lambda\}$ is an operator-bounded approximate identity for A.

((d) $\Rightarrow$ (b)). Suppose that A has an operator-bounded approximate identity $\{u_\lambda\}$. Then standard estimates show that, if $\sum_{k=1}^{\infty}\|a_k\| \, \|f_k\| < +\infty$, then

$$\sum_{k=1}^{\infty} a_k u_\lambda \otimes f_k \to \sum_{k=1}^{\infty} a_k \otimes f_k$$

in $A \mathbin{\hat{\otimes}} A^*$. Consequently, if $\sum_{k=1}^{\infty} f_k * a_k = 0$, then

$$\sum_{k=1}^{\infty} \langle f_k, a_k\rangle = \lim_{\lambda} \sum_{k=1}^{\infty} \langle f_k, a_k u_\lambda\rangle = \lim_{\lambda} \left\langle \sum_{k=1}^{\infty} f_k * a_k, u_\lambda \right\rangle = 0;$$

hence, A has property P2. $\square$

3. A Banach algebra A is said to be modular annihilator if every modular maximal left (right) ideal of A has a nonzero right (left) annihilator (see [**2**]), and is said to be dual if every closed left (right) ideal in A is an annihilator ideal [**11**, Definition 2.8.1]. The former notion is purely algebraic, while the latter is topological; yet, a C^*-algebra is modular annihilator if and only if it is dual [**15**, Theorem 4.1]; [**2**, Example 4.1]. The next theorem shows that the same is true of A^*-algebras of the first kind possessing property P2. If A is a semisimple Banach algebra, then $M_L(A)$ will denote the Banach algebra of (automatically) continuous left multipliers of A. Finally, the bidual $\mathfrak{A}^{**}$ of a C^*-algebra $\mathfrak{A}$ is equipped with a product, extending that of $\mathfrak{A}$, with respect to which it is a von Neumann algebra [**3**, §12.1].

THEOREM 2. *Let A be an A^*-algebra of the first kind with A^2 dense in A, and let $\mathfrak{A}$ be the C^*-algebra completion of A. Then the following statements are equivalent*:

(a) *A is a modular annihilator algebra.*
(b) *$\mathfrak{A}$ is a modular annihilator algebra.*
(c) *A is a dual algebra.*
(d) *$\mathfrak{A}$ is a dual algebra.*
(e) *$M_L(A)$ is algebra isomorphic to $\mathfrak{A}^{**}$.*
(f) *$M_L(\mathfrak{A})$ is algebra isomorphic to $\mathfrak{A}^{**}$.*

PROOF. The implications (c) $\Rightarrow$ (a), (d) $\Rightarrow$ (b), (b) $\Rightarrow$ (a), and (a) $\Rightarrow$ (d) are either immediate or well known (see [**2**], [**15**]). The equivalence (d) $\Leftrightarrow$ (f) may be found in [**10**], and the implication (c) $\Rightarrow$ (e) is the content of [**14**, Theorem 4.2, p. 264]. To complete the proof, the implications (a) $\Rightarrow$ (c), (e) $\Rightarrow$ (f) will be established.

((a) $\Rightarrow$ (c)). This follows from [**13**, Corollary 4.4, p. 426] in view of [**1**, Theorem 4.2, p. 6] (a slip is made in [**1**, Proposition 3.3] in not assuming that $A = \mathfrak{A} \cdot A$).

((e) $\Rightarrow$ (f)). First, since $\mathfrak{A}$ has a bounded approximate identity of norm one, $M_L(\mathfrak{A})$ is isometric and algebra isomorphic to a closed subalgebra of $\mathfrak{A}^{**}$. But the hypothesis (e) implies, since $\mathfrak{A}$ is the closure of A in $\mathfrak{A}^{**}$, that every element of $\mathfrak{A}^{**}$ determines an element of $M_L(\mathfrak{A})$. □

4. If A is a replete Hilbert algebra (for definitions, see [**16**]), then in the so-called Rieffel norm $\|\cdot\|_r$, A is an A^*-algebra of the first kind such that A^2 is dense in A [**5**, Theorem 4.1]. As a result, Theorem 2 applies, and yields the following stronger versions of Theorems 3.7 and 4.3 of [**16**].

PROPOSITION 3. *Let A be a replete Hilbert algebra with C^*-algebra completion $\mathfrak{A}$. If $\mathfrak{A}$ is dual, then A* (*as a Hilbert algebra*) *is dual and has dense socle.*

PROOF. By Theorem 2, the Banach algebra $A_r = (A, \|\cdot\|_r)$ is a dual algebra and, as a result, A_r has dense socle. Since the map $A_r \to A$ is a norm-decreasing bijection, the socle of A is also dense in the Hilbert algebra A. Finally, [**16**, Theorem 3.7] shows that A is a dual Hilbert algebra. $\square$

PROPOSITION 4. *Let A be a replete Hilbert algebra with C^*-algebra completion $\mathfrak{A}$. Then $\mathfrak{A}$ is dual if and only if A is modular annihilator.*

PROOF. Immediate from Theorem 2(a), (d). $\square$

Finally, since every full (i.e., maximal) Hilbert algebra is a replete Hilbert algebra, Proposition 4 can be related to an example in [**8**], where it is shown that the trace class $\tau(A)$ of a full Hilbert algebra A, regarded as a subspace of $\mathfrak{A}^*$, need not be dense in $\mathfrak{A}^*$. In fact, the following is true.

COROLLARY 5. *Let A be a full Hilbert algebra with trace class $\tau(A)$ and C^*-completion $\mathfrak{A}$. Then $\tau(A)$ is dense in $\mathfrak{A}^*$ if and only if A is a modular annihilator algebra.*

PROOF. In [**6**], it is shown that $\tau(A)$ is dense in $\mathfrak{A}^*$ if and only if $\mathfrak{A}$ is dual; hence, the result follows from Proposition 4. $\square$

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