PROCEEDINGS OF THE
AMERICAN MATHEMATICAL SOCIETY
Volume 74, Number 2, May 1979

MULTIPLIERS OF A^*-ALGEBRAS

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ABSTRACT. Let A be an A^*-algebra of the first kind with C^*-algebra completion $\mathfrak{A}$. It is known that if A is dual then A^2 is dense in A and the Banach algebras $M_L(A)$ and $M_L(\mathfrak{A})$ of left multipliers of A and $\mathfrak{A}$ are algebra isomorphic. In this note it is proved that $M_L(A)$ and $M_L(\mathfrak{A})$ are topologically algebra isomorphic when A is an arbitrary A^*-algebra of the first kind such that A^2 is dense in A. As a consequence, it follows that every left multiplier of a replete Hilbert algebra A is automatically continuous.

1. Introduction. Let $(A, \|\cdot\|, |\cdot|)$ be an A^*-algebra where $\|\cdot\|$ is a Banach algebra norm, and $|\cdot|$ is the auxiliary norm [**10**, p. 181]. The Banach algebra $A = (A, \|\cdot\|)$ is of the first kind if A is a *-ideal of its C^*-algebra completion $\mathfrak{A} = (\mathfrak{A}, |\cdot|)$. In this case, since $\|\cdot\|$ majorizes $|\cdot|$ [**10**, Corollary 4.1.16], A is a Banach $\mathfrak{A}$-bimodule (i.e., there exists $K > 0$ such that $\|wa\| \leqslant K|w|\,\|a\|$, $\|aw\| \leqslant K\|a\|\,|w|$, for all $a \in A$, $w \in \mathfrak{A}$ [**3**, Proposition 2.2, Theorem 2.3]). A left multiplier of an algebra B is a linear map $T: B \to B$ such that $T(xy) = (Tx)y$, for all $x, y \in B$; every left multiplier of a semisimple Banach algebra is continuous [**6**]. For A (resp., $\mathfrak{A}$) as above, let $M_L(A)$ (resp., $M_L(\mathfrak{A})$) be the Banach algebra of all (automatically continuous, since $A, \mathfrak{A}$ are semisimple [**10**, Theorem 4.1.19]) left multipliers of A (resp., $\mathfrak{A}$). Now, if A is a dual A^*-algebra of the first kind, then $A^2 = \operatorname{sp}\{ab: a, b \in A\}$ is dense in A [**10**, Corollary 2.8.3], and $M_L(A)$, $M_L(\mathfrak{A})$ are algebra isomorphic [**12**, Theorem 5.1], [**11**, Theorem 4.2]. In this paper, we show that if A is an arbitrary A^*-algebra of the first kind with A^2 dense in A, then $M_L(A)$ is topologically algebra isomorphic to $M_L(\mathfrak{A})$. As an application of this result, it is proved that every left multiplier of a replete Hilbert algebra $A = (A, \|\cdot\|)$ is automatically continuous. This result appears to be new even for full Hilbert algebras. In general, the Hilbert space norm $\|\cdot\|$ is not an algebra norm on A, nor is A complete in this norm. The authors know of no other example of an automatic continuity result for multipliers of a non-Banach nonnormed algebra.

2. Main result. As in [**3**], we denote the spectrum of an element x in a Banach algebra B by $\operatorname{Sp}_B(x)$ and its spectral radius by $\nu_B(x)$.

THEOREM 1. *Let A be an A^*-algebra of the first kind with A^2 dense in A, and*

Received by the editors August 29, 1978.
AMS (MOS) subject classifications (1970). Primary 46L04, 46L20; Secondary 46H25, 46K15.
Key words and phrases. Multiplier, A^*-algebra, C^*-algebra, Hilbert algebra, automatic continuity.

let $\mathfrak{A}$ be the C^*-algebra completion of A. Then $M_L(A)$ is topologically algebra isomorphic to $M_L(\mathfrak{A})$.

PROOF. First, since A^2 is dense in A, the Hewitt-Cohen factorization theorem [5, Theorem 32.22] implies that $A = \mathfrak{A} \cdot A = \{wa\colon w \in \mathfrak{A}, a \in A\}$. Thus, if $T \in M_L(\mathfrak{A})$, then $T' = T|_A$ maps A into A (since $T'(A) = T(A) = T(\mathfrak{A} \cdot A) = T(\mathfrak{A}) \cdot A \subseteq \mathfrak{A} \cdot A = A$); hence, $T' \in M_L(A)$. Next, because A is a dense $*$-ideal of $\mathfrak{A}$ and $\mathfrak{A} \cdot A = A$, A possesses a (left) approximate identity $\{u_\alpha\}_\alpha$ such that $|u_\alpha| \leqslant 1$, for all α [4, Proposition 1.7.2]. Therefore, for each a in A,

$$\begin{aligned} \|T'a\| &= \lim_\alpha \|T'(u_\alpha a)\| = \lim_\alpha \|(Tu_\alpha)a\| \\ &\leqslant \sup_\alpha K|Tu_\alpha| \, \|a\| \leqslant K\|T\| \, \|a\|; \end{aligned}$$

whence $\|T'\| \leqslant K\|T\|$. Now, A is dense in $\mathfrak{A}$, so the continuous linear map $T \mapsto T'$ from $M_L(\mathfrak{A})$ into $M_L(A)$ is a vector space isomorphism, and is easily seen to be an algebra isomorphism as well. Thus, it remains only to show that the map $T \mapsto T'$ is surjective.

Since A is a left ideal of $\mathfrak{A}$ and of $M_L(A)$, $\mathrm{Sp}_{\mathfrak{A}}(a) \cup \{0\} = \mathrm{Sp}_{M_L(A)}(a) \cup \{0\}$ [3, proof of Proposition 4.1]; hence $\nu_{\mathfrak{A}}(a) = \nu_{M_L(A)}(a)$, for each a in A. Consequently, if L_a denotes left multiplication by a in $M_L(A)$, then

$$\begin{aligned} |a|^2 = |a^*a| = \nu_{\mathfrak{A}}(a^*a) &= \nu_{M_L(A)}(a^*a) \leqslant \|L_{a^*a}\| \\ &= \sup\{\|a^*ab\|\colon b \in A, \|b\| \leqslant 1\} \\ &\leqslant K|a^*| \sup\{\|ab\|\colon b \in A, \|b\| \leqslant 1\} = K|a| \, \|L_a\|, \end{aligned}$$

and so $|a| \leqslant K\|L_a\| \leqslant K^2|a|$, for all a in A. Now, let $S \in M_L(A)$ be given. Then, for each a in A,

$$|Sa| \leqslant K\|L_{Sa}\| = K\|SL_a\| \leqslant K\|S\| \, \|L_a\| \leqslant K^2\|S\| \, |a|;$$

therefore, S extends uniquely to a continuous linear operator T on $\mathfrak{A}$ with $\|T\| \leqslant K^2\|S\|$. It follows immediately that $T \in M_L(\mathfrak{A})$ and that $T' = S$. $\square$

Observe that if the A^*-algebra A is an isometric Banach $\mathfrak{A}$-bimodule (i.e., if the constant $K = 1$), then $M_L(A)$ is isometrically algebra isomorphic to $M_L(\mathfrak{A})$. In addition, under the assumptions of Theorem 1, it follows mutatis mutandis that the Banach algebras $M_R(A)$, $M_R(\mathfrak{A})$ of all right multipliers of A, $\mathfrak{A}$ are topologically algebra isomorphic. This fact, together with Theorem 1, implies that the Banach algebras $M(A)$, $M(\mathfrak{A})$ of all double multipliers of A, $\mathfrak{A}$ are also topologically algebra isomorphic.

3. Application. If $A = (A, \|\cdot\|)$ is a replete Hilbert algebra (in particular, every full Hilbert algebra is replete; see [7], [13] for definitions), then in the so-called Rieffel norm $\|\cdot\|_r$, $A_r = (A, \|\cdot\|_r)$ is an A^*-algebra of the first kind such that A_r^2 is dense in A_r [7, Theorem 4.1]. Thus, Theorem 1 applies, and yields the following result. Let $M_L(A)$ be the Banach algebra of all continuous left multipliers of $A = (A, \|\cdot\|)$.

THEOREM 2. *If A is a replete Hilbert algebra with C^*-algebra completion $\mathfrak{A} = (\mathfrak{A}, |\cdot|)$, and $A_r = (A, \|\cdot\|_r)$, then $M_L(A) = M_L(A_r)$ as Banach algebras. Hence, every left multiplier of A is automatically continuous.*

PROOF. First, since $A = A_r$ as abstract algebras, and A_r is a semisimple Banach algebra, $M_L(A) \subseteq M_L(A_r)$ set-theoretically. On the other hand, A_r is an isometric Banach $\mathfrak{A}$-bimodule; hence, $M_L(A_r)$ is isometrically algebra isomorphic to $M_L(\mathfrak{A})$ by Theorem 1. Further, since $\mathfrak{A} \subseteq \mathbf{B}(H)$, the bounded linear operators on the Hilbert space completion $H = (H, \|\cdot\|)$ of $A = (A, \|\cdot\|)$, it follows from [**1**, Proposition 4.2] that $M_L(\mathfrak{A})$ is isometrically algebra isomorphic to a closed subalgebra of $\mathbf{B}(H)$. Hence, every left multiplier T in $M_L(A_r)$ is continuous on $A = (A, \|\cdot\|)$ (i.e., $M_L(A_r) \subseteq M_L(A)$) and, in addition, the Banach algebras $M_L(A)$ and $M_L(A_r)$ have the same norm. Finally, if T is a left multiplier of $A = A_r$, then (since A_r is semisimple) $T \in M_L(A_r) = M_L(A)$. $\square$

In an earlier manuscript [**8**], the authors gave a proof of Theorem 2 in the spirit of [**9**], [**2**]. We would like to thank G. F. Bachelis for his helpful comments regarding [**8**], consideration of which eventually led to Theorem 1 in its present generality.

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