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P-POINTS IN RANDOM UNIVERSES

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ABSTRACT. A pathway is defined as an increasing sequence of subsets of ${}^{\omega}\omega$ which satisfy certain closure and boundedness properties. The existence of a pathway is shown to imply the existence of a P-point in $\beta N \setminus N$. Pathways are shown to exist in any random extension of a model of ZFC + CH.

A point in a topological space is called a P-point if the intersection of any countable family of its neighborhood is a neighborhood. Shelah **[8]** has recently shown the existence of a P-point in $\beta N \setminus N$ to be independent of ZFC.

Various assumptions are known to imply the existence of P-points **[1]**, **[4]**, **[6]**. In this paper we contribute a new axiom of this sort and we show that the axiom holds in any random extension of a model of ZFC + CH. The referee has pointed out that Kunen has shown the existence of P-points in certain models of this description **[5]**.

1. From pathways to P-points. If $f, g \in {}^{\omega}\omega$ then $f \leqslant g$ is taken to mean that $\{n | f(n) > g(n)\}$ is finite. It will be convenient to identify a subset of ω with its characteristic function. Thus if $a, b \subseteq \omega$ then $a \leqslant b$ means that $a - b$ is finite.

An unbounded $f \in {}^{\omega}\omega$ can be interpreted as a sequence

$$\omega = f^{(0)} \supseteq f^{(1)} \supseteq \cdots$$

where $f^{(n)} = \{k \in \omega | f(k) < n\}$. A free ultrafilter U on ω is called a P-point provided that whenever $f \in {}^{\omega}\omega$ is unbounded and $f^{(n)} \in U$ for all $n \in \omega$ there is an $a \in U$ such that $a \leqslant f^{(n)}$ for all $n \in \omega$. A discussion of the relationship of this definition to βN can be found in **[7]**.

Let κ be the smallest cardinal of a $\leqslant$-dominating subset of ${}^{\omega}\omega$. Ketonen **[4]** has shown that if $\kappa = 2^{\omega}$ then there is a P-point. What follows is a refinement of his construction.

Call a sequence $\langle A_\alpha | \alpha < \kappa \rangle$ a *pathway* provided:

(a) $\bigcup_{\alpha<\kappa} A_\alpha = {}^{\omega}\omega$,

(b) $A_\alpha \subseteq A_\beta$ whenever $\alpha < \beta$,

(c) A_α does not $\leqslant$-dominate $A_{\alpha+1}$,

(d) $(f \text{ join } g) \in A_\alpha$ whenever $f, g \in A_\alpha$ (where $(f \text{ join } g) = h \in {}^{\omega}\omega$ is defined for $n \in \omega$ by $h(2n) = f(n)$ and $h(2n + 1) = g(n)$),

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(e) if $f \leqslant_T g$ and $g \in A_\alpha$ then $f \in A_\alpha$.The symbol $\leqslant_T$ is used for Turing reducibility.

1.1 THEOREM. *The existence of a pathway implies the existence of a P-point.*

PROOF. We begin by listing some easily derived properties of the pathway $\langle A_\alpha | \alpha < \kappa \rangle$.

(i) $a \cap b \in A_\alpha$ whenever $a, b \in A_\alpha$,

(ii) if $f \in A_\alpha$ is unbounded then there is a $g \in A_\alpha$ where for all $n \in \omega$, $f^{(n)}$ has at least n members smaller then $g(n)$,

(iii) if $f, g \in A_\alpha$ and f is unbounded then $\binom{f}{g} \in A_\alpha$ where

$$\binom{f}{g} = \{k \in \omega | \exists n < f(k)\ [k < g(n)]\},$$

(iv) if $f, g \in A_\alpha$ then $h \in A_\alpha$ where for $n \in \omega$, $h(n) = \min(f(n), g(n))$,

(v) if $f, g \in A_\alpha$ and $g \in {}^\omega 2$ then $h \in A_\alpha$ where $h(n) = f(n) \cdot g(n)$.

Let us say that $U \subseteq {}^\omega 2$ is a free filterbase if whenever $a, b \in U$ there is an infinite $c \in U$ with $c \subseteq a \cap b$.

By transfinite recursion we choose a sequence,

$$U_0 \subseteq U_1 \subseteq \cdots \subseteq U_\alpha \subseteq \cdots \quad (\alpha < \kappa)$$

where U_α is maximal among the free filterbases $X \subseteq A_\alpha \cap {}^\omega 2$. If U_α has been chosen then by c there is an $f_\alpha \in A_{\alpha+1}$ such that for no $g \in A_\alpha$ is $f \leqslant g$. Provided that

$$U_\alpha^* = U_\alpha \cup \left\{ \binom{g}{f_\alpha} \Big| g \in A_\alpha \text{ is unbounded} \wedge \forall n g^{(n)} \in U_\alpha \right\}$$

is a free filterbase we require $U_{\alpha+1} \supseteq U_\alpha^*$. Notice that (iii) guarantees that $U_\alpha^* \subseteq A_{\alpha+1}$. We will show that $U = \bigcup_{\alpha<\kappa} U_\alpha$ is a P-point.

It must first be verified that each U_α^* is indeed a free filterbase; this will follow from I, II and III.

I. If $\binom{g}{f_\alpha} \in U_\alpha^*$, where $\forall n g^{(n)} \in U_\alpha$ then by (ii) there is an $h \in A_\alpha$ such that $g^{(n)}$ always has more than n members smaller than $h(n)$. By the choice of f_α we have $f_\alpha \not\leqslant h$ and so $c = \{n | f_\alpha(n) > h(n)\}$ is infinite. But then

$$\binom{g}{f_\alpha} = \bigcup_{n\in\omega} \{k < f_\alpha(n) | k \in g^{(n)}\}$$

must be infinite since whenever $n \in c$

$$\{k < f_\alpha(n) | k \in g^{(n)}\} \supseteq \{k < h(n) | k \in g^{(n)}\}$$

has at least n members.

II. Suppose $g_1, g_2 \in A_\alpha$ are unbounded and $\forall n g_1^{(n)} \cap g_2^{(n)} \in U_\alpha$. By (iv), $g \in A_\alpha$ where

$$g(n) = \min(g_1(n), g_2(n)).$$

For $n \in \omega$, $g^{(n)} = g_1^{(n)} \cap g_2^{(n)}$. It follows that $\binom{g}{f_\alpha} \in U_\alpha \in U_\alpha^*$ and that

$$\binom{g}{f_\alpha} = \{k \in \omega | \exists n < g(k)\ [k < f_\alpha(n)]\} \subseteq \binom{g_1}{f_\alpha} \cap \binom{g_2}{f_\alpha}.$$

III. Suppose $a, g \in A_\alpha$, $a \in {}^\omega 2$, g is unbounded and $\forall n g^{(n)} \cap a \in A_\alpha$. By (v), $h \in A_\alpha$ where $h(n) = f(n) \cdot g(n)$. For $n \in \omega$, $h^{(n)} = g^{(n)} \cap a$. Thus

$$\binom{g}{f_\alpha} \cap a = \binom{h}{f_\alpha} \in U_\alpha^*.$$

Since each U_α is a free filterbase so is U. If $a \subseteq \omega$ then for some α $a \in A_\alpha$. Since either $a \in U_\alpha$ or $\omega - a \in U_\alpha$ we have that U is an ultrafilter.

Suppose $g \in {}^\omega\omega$ is unbounded and such that $\forall n \in \omega g^{(n)} \in U$. For some α, $g \in A_\alpha$ and so $\forall n g^{(n)} \in U_\alpha$. But then $\binom{g}{f_\alpha} \in U_{\alpha+1} \subseteq U$ and for all $n \in \omega$, $\binom{g}{f_\alpha} \subseteq g^{(n)}$. Thus U is a P-point.

2. Pathways in random universes. Let $\mathfrak{M}$ be a countable standard transitive model of ZFC in which λ is an infinite ordinal number. Let $R = R_\lambda$ be the cartesian product in $\mathfrak{M}$ of λ copies of 2 endowed with the product measure. In $\mathfrak{M}$, let $B = B_\lambda$ be the Boolean algebra of measurable sets modulo the sets of measure zero. Since B is c.c.c. and countably complete it is complete in $\mathfrak{M}$.

If H is B-generic over $\mathfrak{M}$ and $r \in {}^\lambda 2$ is such that for $\alpha < \lambda$, $r(\alpha) = 1$ iff $\{f \in R \mid f(\alpha) = 1\}$ is in a member of H, then r is called random over $\mathfrak{M}$.

2.1 LEMMA. *If G is B-generic over $\mathfrak{M}$ then ${}^\omega\omega \cap \mathfrak{M}$ dominates ${}^\omega\omega \cap \mathfrak{M}[G]$.*

Lemma 2.1 is proven in **[10]**. The development of Lemmas 2.2, 2.3 and 2.4 can be quite similar to that found in **[9]** (special care must be taken with the absoluteness arguments when λ is uncountable). At the suggestion of the referee, the proofs of these lemmas are left as exercises for the reader.

2.2 LEMMA. *Suppose $\mathfrak{M} \supseteq \mathfrak{N}$ where $\mathfrak{M}$ is a countable standard transitive model of ZFC. If $r \in {}^\lambda 2$ is random over $\mathfrak{M}$ then it is also random over $\mathfrak{N}$.*

2.3 LEMMA. *If $r \in {}^\lambda 2$ is random over $\mathfrak{M}$ and $\Gamma \in ({}^\omega\lambda \cap \mathfrak{M})$ is injective then $r \circ \Gamma \in {}^\omega 2$ is random over $\mathfrak{M}$.*

2.4 LEMMA. *Suppose $\mathfrak{M}$ is a model of ZFC such that for each $s \in {}^\omega 2 \cap \mathfrak{M}$ there is an $\alpha < \omega_1$ such that $L_\alpha[s]$ is a model of ZFC. If $r \in {}^\lambda 2$ is random over $\mathfrak{M}$ then for every $t \in {}^\omega 2 \cap \mathfrak{M}[r]$ there is an $\alpha < \omega_1$, an $s \in {}^\omega 2 \cap \mathfrak{M}$ and an injective $\Gamma \in {}^\omega\lambda \cap \mathfrak{M}$ such that $t \in L_\alpha[r \circ \Gamma, s]$.*

2.5 THEOREM. *If $\mathfrak{M}$ is a model of ZFC + CH, ν is an infinite ordinal of $\mathfrak{M}$ and $r \in {}^\nu 2$ is random over $\mathfrak{M}$ then there is a pathway in $\mathfrak{M}[r]$. By Theorem* 1.1 *it follows that there is a P-point in $\mathfrak{M}[r]$.*

PROOF. We first observe that we may weaken clause c in the definition of a pathway to the statement that "A_α does not dominate ${}^\omega\omega$."

In Lemma 2.4 it was assumed that the $L_\alpha[s]$ were models of ZFC. For the result to hold, however, it is necessary only that the $L_\alpha[s]$ be models of some finite fragment Φ_1 of ZFC. Similarly, by Lemma 2.1 there is a finite fragment $\Phi \supseteq \Phi_1$ of ZFC such that whenever $L_\alpha[s]$ is a model of Φ and $r \in {}^\omega 2$ is random over $L_\alpha[s]$ then ${}^\omega\omega \cap L_\alpha[s]$ $\leqslant$-dominates ${}^\omega\omega \cap L_\alpha[r, s]$. For each $s \in {}^\omega 2 \cap \mathfrak{M}$, we know by the reflection principle that there are arbitrarily large $\alpha < \omega_1$ for which $L_\alpha[s]$ is a model of Φ.

In $\mathfrak{M}$, choose a sequence $\langle s_\alpha | \alpha < \omega_1\rangle$ from ${}^\omega 2$ and an increasing sequence $\langle \xi_\alpha | \alpha < \omega_1\rangle$ from ω_1 such that if $\mathfrak{M}_\alpha$ is defined as $L_{\xi_\alpha}[s_\alpha]$ then

A. $\mathfrak{M}_\alpha$ is a model of Φ,

B. $\alpha < \beta < \omega_1$ implies $\mathfrak{M}_\alpha \subseteq \mathfrak{M}_\beta$,

C. ${}^\omega 2 \cap \mathfrak{M} = \bigcup_{\alpha<\omega_1}({}^\omega 2 \cap \mathfrak{M}_\alpha)$.

Let $\mathfrak{G}$ be the set of all injective $\Gamma \in {}^\omega\nu \cap \mathfrak{M}$ and for $\alpha < \omega_1$ define

$$A_\alpha = \bigcup \left\{{}^\omega\omega \cap \mathfrak{M}_\alpha[r \circ \Gamma] | \Gamma \in \mathfrak{G}\right\}.$$

It is easy to see that A_α is closed under Turing reductions and finite joints. Since the (in $\mathfrak{M}[r]$) countable set ${}^\omega\omega \cap \mathfrak{M}_\alpha$ $\leqslant$-dominates A_α it is clear that A_α cannot dominate ${}^\omega\omega \cap \mathfrak{M}[r]$. Finally, since by 2.4,

$$\bigcup_{\alpha<\omega_1} A_\alpha = {}^\omega\omega \cap \mathfrak{M}[r]$$

it follows that $\langle A_\alpha | \alpha < \omega_1^{[r]}\rangle$ is a pathway in $\mathfrak{M}[r]$.

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