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EQUIVALENCE OF NORMS FOR COEFFICIENTS OF UNITARY GROUP REPRESENTATIONS[1]

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ABSTRACT. We study the equivalence between the $B(G)$ norm and the supremum norm in the Banach space A_π, the closed subspace of $B(G)$ which is generated by the coefficients of a continuous, unitary, irreducible, infinite dimensional representation π of a locally compact, noncompact group G.

1. Introduction. Let $B(G)$ be the Fourier-Stieltjes algebra of a locally compact group G with the usual norm:

$$\|u\|_B = \sup \left| \int_G f(x)u(x)\,dx \right| : f \in L^1(G), \qquad \|f\|_{C^*(G)} \leqslant 1,$$

$C^*(G)$ being the C^* algebra of the group G; let A_π be the closed subspace of $B(G)$ generated by the coefficients $\{(\pi(x)w, z)\colon w, z \in H_\pi\}$ of a unitary representation π of G on a Hilbert space H_π with inner product $(\cdot, \cdot)$ **[1]**, **[2]**, **[3]**.

We are interested in comparing the B norm and the supremum norm on this subspace. If the representation π is contained in the left regular representation of G and G is abelian and compact, then the equivalence of the two norms on A_π is equivalent to the support of π being a Sidon set in the dual group of G [**4**, Theorem 5.7.3], [**2**, (2.13)]. Sidon sets have also been studied in the context of nonabelian compact groups.

In the noncompact, nonabelian case, even the problem whether for a single infinite dimensional irreducible representation the two norms are equivalent or not on A_π seems to be a nontrivial one. Although we are not able to give any necessary and sufficient condition, we prove that for a fairly large class of such representations the norms are inequivalent.

On the other hand we give a sufficient condition for the equivalence, and we provide an example in which this phenomenon occurs. Our results are contained in the following theorems:

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THEOREM 1. *If the subgroup $\pi(G)$ of the unitary group $\mathcal{U}$ on H_π satisfies the condition*:

> "*there exists a $\delta > 0$ such that for every two finite orthonormal systems $\{w_i\}$, $\{z_i\}$, $(i = 1, \ldots, k)$ in H_π and for every unitary operator $U \in \mathcal{U}$, there exists an operator $U' \in \pi(G)$ such that*:
>
> $$|((U - U')w_i, z_i)| < (1 - \delta) \qquad (i = 1, \ldots, k)\text{''}$$

then the B norm and the supremum norm are equivalent on A_π.

THEOREM 2. *If all the coefficients of the representation π vanish at infinity* (*possibly modulo the kernel of the representation*), *then the norms are inequivalent on A_π.*

THEOREM 3. *Let π be a representation of G which is induced by a representation ρ of a closed subgroup N such that the homogeneous space G/N is not discrete. Assume that*:
either G/N has a G-invariant measure,
or N is compact and G/N has a relatively invariant measure;
then the two norms are inequivalent in A_π.

We recall that the representation π is always assumed irreducible. For theorems and terminology about induced representations we will refer to **[5]**.

We denote by $\mathcal{L}(H)$ the algebra of all the bounded linear operators on a Hilbert space H, and by $\mathcal{T}(H)$ the ideal of the trace class operators with norm: $\|T\|_1 = \mathrm{tr}(|T|)$, where $|T|$ is the absolute value of T and tr is the canonical, faithful, normal semifinite trace on $\mathcal{L}(H)^+$. We recall that $\mathcal{L}(H)$ can be identified with the dual of $\mathcal{T}(H)$ via the pairing

$$\langle L, T\rangle = tr(LT), \quad L \in \mathcal{L}(H), T \in \mathcal{T}(H)\ [\mathbf{6}].$$

We will make use of the following characterization of A_π, for π irreducible, which can be easily deduced from [**7**, Theorem 3].

PROPOSITION 1. *A_π is isometrically isomorphic to $\mathcal{T}(H_\pi)$ via the so-called* "*inverse Fourier transform localized at π*":

$$t(x) = \mathrm{tr}(\pi(x)T), \qquad x \in G, T \in \mathcal{T}(H_\pi);$$

i.e. every function in A_π is of the form t for some $T \in \mathcal{T}(H_\pi)$, the correspondence is one-to-one and $\|t\|_{B(G)} = \|T\|_1$.

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2. Proof of Theorem 1. We prove a more general result for any subgroup $\mathcal{V}$ of the unitary group $\mathcal{U}$ of the Hilbert space H_π satisfying the hypothesis of the theorem. Namely, if for such subgroup we define

$$n(T) = \sup|\mathrm{tr}(UT)|\colon U \in \mathcal{V},$$

then

$$\delta\|T\|_1 \leqslant n(T) \leqslant \|T\|_1, \quad \text{for every } T \in \mathfrak{T}(H_\pi).$$

The theorem will follow by Proposition 1, choosing $\mathcal{V} = \pi(G)$ so that $n(T) = \|t\|_\infty$.

It is enough to prove the left-hand side inequality for all the finite rank operators, which are dense in $\mathfrak{T}(H_\pi)$ with respect to the trace norm. The right-hand side inequality is trivial. Without loss of generality let the operator T be defined by:

$Tw = \sum_{i=1}^k \lambda_i(w, z_i)w_i$, $w \in H_\pi$, λ_i complex numbers, $\{w_i\}$, $\{z_i\}$ being orthonormal systems in H_π.

Then we have

$$T^* = \sum_{i=1}^{k} \bar{\lambda}_i(\cdot, w_i)z_i,$$

hence

$$|T| = (T^*T)^{1/2} = \sum_{i=1}^{k} |\lambda_i|(\cdot, z_i)z_i$$

and $T = V|T|$, where V is the partial isometry defined by:

$$z_i \to (\lambda_i/|\lambda_i|)w_i, \qquad i = 1, \ldots, k,$$

which can be extended to a unitary operator V'' on H_π. Then $T = V|T| = V''|T|$, since

$$\text{range}(|T|) \subset \text{span}(\{z_i\}),$$

so that we have $|T| = UT$, with U unitary, $U = (V'')^*$, adjoint of V''. Now if $\{z_i\}$ $(i = 1, 2, \ldots, \infty)$ is a completion of the orthonormal system $\{z_i\}$ $(i = 1, \ldots, k)$, one has, for every $W \in \mathfrak{U}$:

$$\begin{aligned}
\text{tr}(WT) &= \sum_{j=1}^{\infty} (WTz_j, z_j) \\
&= \sum_{j=1}^{\infty} \left(W\left[\sum_{i=1}^{k} \lambda_i(z_j, z_i)w_i \right], z_j \right) \\
&= \sum_{j=1}^{k} (W\lambda_j w_j, z_j) = \sum_{j=1}^{k} \lambda_j(Ww_j, z_j).
\end{aligned}$$

On the other hand it is easy to see that:

$$\text{tr}(|T|) = \sum_{i=1}^{k} |\lambda_i| = \|T\|_1.$$

Let $U' \in \mathcal{V}$ be such that: $|([U - U']w_i, z_i)| < 1 - \delta$, $i = 1, \ldots, k$. We get

$$\{\sup|\operatorname{tr}(WT)|: W \in \mathcal{V}\} = n(T) \geqslant |\operatorname{tr}(U'T)|$$
$$= \left|\sum_{i=1}^{k} \lambda_i([U' - U]w_i, z_i) + \sum_{i=1}^{k} \lambda_i(Uw_i, z_i)\right|$$
$$\geqslant \left| \|T\|_1 - \sum_{i=1}^{k} |\lambda_i| \, |([U' - U]w_i, z_i)| \right|$$
$$\geqslant \left| \|T\|_1 - (1 - \delta) \sum_{i=1}^{k} |\lambda_i| \right| = \delta \|T\|_1.$$

EXAMPLE 1. Let $\mathcal{U}$ be the discrete unitary group on the (separable) Hilbert space H, and let π be the standard representation of $\mathcal{U}$ onto itself; π is clearly irreducible, continuous, infinite dimensional if H is so, and $\pi(\mathcal{U}) = \mathcal{U}$ satisfies trivially the hypothesis of Theorem 1 for every $\delta > 0$, so that the two norms are equal on A_π.

REMARK 1. In [**8**, Theorem 4] the closed normal subgroups of the unitary group of a Hilbert space H are characterized. Except for the finite subgroups, they have to contain all those unitary operators which act as the identity on the complement of a finite dimensional subspace, so that they have to be weakly operator dense in the unitary group. Consequently they satisfy the hypothesis of Theorem 1 as well, and the same kind of argument applies to their standard representation in the unitary group as in Example 1.

3. Proof of Theorem 2. In this section we prove Theorem 2, i.e. the inequivalence of the two norms on A_π, for a representation π whose coefficients vanish at infinity.

This class includes, for instance, the square integrable representations, since their coefficients may be written as convolutions of two functions in $L^2(G)$ [**6**, (14.1)].

LEMMA 1. *Let K be a compact subset in a Hilbert space H, $\{w_n\}$ an infinite orthonormal system; then*

$$\lim_{n \to +\infty} \{\sup|(z, w_n)|: z \in K\} = 0.$$

PROOF. By standard arguments.

LEMMA 2. *Let $\{u_n(x)\}$ be a sequence of complex continuous functions on a locally compact space X such that*:

(i) $|u_n(x)| \leqslant 1$ *for every* $x \in X$, $n = 0, 1, 2, \ldots$,

(ii) u_n *vanishes at infinity for every* n,

(iii) $\{u_n\}$ *tends to zero uniformly on compact sets as* $n \to +\infty$.

Then, given $\varepsilon > 0$ *and a sequence* $\{c_i\} \in l^2$ *such that*

$$\|c_i\|_\infty < \varepsilon/2\|c_i\|_2$$

there exists a subsequence $\{u_{n_i}\}$ *such that*

$$\left\| \sum_{i=0}^{\infty} c_i u_{n_i}(x) \right\|_\infty \leqslant \varepsilon \|c_i\|_2.$$

Here, as usual, $\|c_i\|_\infty = \sup|c_i|$; $\|c_i\|_2 = \{\sum_{i=0}^{\infty}|c_i|^2\}^{1/2}$.

PROOF. Choose a monotone sequence of positive numbers $\{\varepsilon_i\}$ such that $\|\varepsilon_i\|_2 < \varepsilon/2$. Define $n_0 = 0$, and

$$K_0 = \left\{x \in X\colon \left|u_{n_0}(x)\right| \geqslant \varepsilon_0\right\};$$

let n_1 be such that, if $n \geqslant n_1$, then $|u_n(x)| < \varepsilon_1$ for $x \in K_{n_0}$. In general let n_i be such that, for $n \geqslant n_i$,

$$|u_n(x)| < \varepsilon_i \quad \text{for every } x \in (K_0 \cup K_1 \cup \cdots \cup K_{i-1})$$

and

$$K_i = \left\{x \in X\colon \left|u_{n_i}(x)\right| \geqslant \varepsilon_i\right\}.$$

Notice that the K_i's are compact by (ii) and disjoint; for, if $x \in K_j$, then $|u_{n_j}(x)| \geqslant \varepsilon_j$, hence x cannot belong to K_i for $i < j$. Now, if $x \notin \bigcup_{i=0}^{\infty} K_i$, then $|u_{n_i}(x)| < \varepsilon_i$ for every i, and

$$\left|\sum_{i=0}^{\infty} c_i u_{n_i}(x)\right| \leqslant \sum_{i=0}^{\infty} |c_i|\,|u_{n_i}(x)| \leqslant \sum_{i=0}^{\infty} \varepsilon_i c_i \leqslant \|\varepsilon_i\|_2\|c_i\|_2 \leqslant \varepsilon/2\|c_i\|_2.$$

If $x \in \bigcup_{i=0}^{\infty} K_i$, then there exists exactly one j such that $x \in K_j$; then, by (i):

$$\left|\sum_{i=0}^{\infty} c_i u_{n_i}(x)\right| \leqslant \left\{\sum_{i\neq j} |c_i|\,|u_{n_i}(x)| + |c_j|\,|u_{n_j}(x)|\right\}$$
$$\leqslant \varepsilon/2\|c_i\|_2 + \varepsilon/2\|c_i\|_2.$$

PROOF OF THEOREM 2. Given $\varepsilon > 0$, choose an infinite orthonormal system $\{w_n\}$ and a sequence of numbers $\{c_i\}$ as in Lemma 2. Define

$$u_n(x) = (\pi(x)w_n, w_0).$$

It is straightforward to see that the sequence $\{u_n\}$ satisfies (i) and (ii) of Lemma 2, while (iii) follows from the identity

$$(\pi(x)w_n, w_0) = (w_n, \pi^*(x)w_0),$$

Lemma 1 and the continuity of π.Therefore we may find a subsequence $\{u_{n_i}\}$ satisfying the conclusion of Lemma 2. Define the operator T by:

$$T = \sum_{i=0}^{\infty} c_i(\cdot, w_0)w_{n_i}.$$

An easy computation shows that $\operatorname{tr}(|T|) = \|c_i\|_2$. Moreover:

$$u(x) = t(x) = \operatorname{tr}(\pi(x)T) = \sum_{i=0}^{\infty} c_i(\pi(x)w_{n_i}, w_0).$$

Then by Lemma 2

$$\|u\|_\infty \leqslant \varepsilon\|c_i\|_2 = \varepsilon \operatorname{tr}(|T|) = \varepsilon\|u\|_B.$$

Since ε is arbitrary the two norms are inequivalent.

REMARK 2. This argument can actually be carried out whenever we have a family $\mathcal{G}$ of subsets of G such that:

(i) $\mathcal{G}$ is stable under finite unions,

(ii) there exists an orthonormal system $\{w_n\}$ in the Hilbert space H_π such that the sequence of coefficients

$$u_n(x) = (\pi(x)w_n, w_0)$$

goes to zero uniformly on the sets of $\mathcal{G}$,

(iii) the coefficients $u_n(x)$ "vanish at infinity" with respect to $\mathcal{G}$, i.e. for every $\varepsilon > 0$ the set $\{x \in G: |u_n(x)| \geqslant \varepsilon\}$ belongs to $\mathcal{G}$.

REMARK 3. If we replace the topology of G with a finer one (e.g. the discrete topology) we may get a representation whose coefficients do not vanish at infinity, while the two norms are still inequivalent, for A_π does not change [**2**, (2.11)], [**1**].

REMARK 4. By the functorial properties of the space A_π, [**2**, (2.10)] (see also [**1**, (2.20)]), Theorem 2 holds for representations whose coefficients vanish at infinity modulo the kernel.

4. Proof of Theorem 3. (We assume, for simplicity, the separability of the Hilbert space H_ρ; the group G is not necessarily separable.)

Let us recall that the space H_π of the induced representation π is the space of (the equivalence classes of) the functions $\phi: G \to H_\rho$ such that:

(i) $\phi(xn) = \rho^*(n)\phi(x)$ for every $x \in G$, $n \in N$,

(ii) $(\phi(x), z)$ are measurable functions of x for every $z \in H_\rho$,

(iii)
$$\|\phi\|_{H_\pi} = \left\{ \int_{G/N} \|\phi(x')\|^2_{H_\rho} \, d\mu(x') \right\}^{1/2} < +\infty,$$

x' being the equivalence class of x in G/N and μ the relatively invariant measure on G/N [**5**, Chapters V, VI].

The inner product in the space H_π is

$$(\phi, \psi) = \int_{G/N} (\phi(x), \psi(x))_{H_\rho} \, d\mu(x').$$

If $\lambda(g)$, $g \in G$, is the modular function of the relatively invariant measure μ on G/N, we have:

$$[\pi(g)\phi](x) = \phi(g^{-1}x)\{\lambda(g^{-1})\}^{1/2}.$$

Let us consider a continuous, nonnegative real valued function f on G, whose support S is such that $S \cap CN$ is compact for every compact C in G, and which satisfies the relation

$$\int_N f(xn) \, dn = 1 \quad \text{for every} x \in G,$$

dn being the left Haar measure on N [**5**, p. 258].

Let the function $f^\#: G \to H_\rho$ be defined by:

$$f^\#(x) = \int_N \rho(n) w f(xn) \, dn,$$

w being a fixed unitary vector in H_ρ; then $f^\#$ is continuous, satisfies condition (i), and $\|f^\#(x)\|_{H_\rho} \leqslant 1$ for every $x \in G$. This follows from [**5**, Lemma 3, p. 260] and [**5**, Lemma 2, p. 372] combined. Moreover $f^\#(e) = \hat{\rho}(f|_N)w$, where e

is the unit of G and $\rho^\wedge$ is the representation of $L^1(N)$ canonically associated to the representation ρ of N; therefore the three above properties of $f^\#$ ensure that, if $S \cap N$ is chosen small enough, then $f^\#$ is different from zero in an open saturated neighborhood of e. Let $\{E_j\}, j = 0, 1, \ldots,$ be a sequence of compact sets in G/N with the following properties:

(a) $\mu(E_j) \to 0$ as $j \to +\infty$,

(b) $E_0 \cap E_j = \varnothing$ and $E_{j+1} \subset E_j$ for $j = 1, 2, \ldots,$

(c) the saturated subsets of G: $F_j = p^{-1}(E_j)$ are contained in the open saturated set in G where the function $f^\#$ is not zero. (p denotes the projection $G \to G/N$.)

If we define

$$\phi_j(x) = \left\{ \int_{E_j} \|f^\#(x')\|^2 d\mu(x') \right\}^{-1/2} \chi_j(x) f^\#(x)$$

where χ_j is the characteristic function of the set F_j, it is easy to see that the functions ϕ_j belong to the space H_π and that $\|\phi_j\| = 1$. If T_j is the rank one operator on H_π defined by:

$$T_j(\psi) = (\psi, \phi_0)\phi_j,$$

then $\|t_j\|_B = \|T_j\| = 1$ for every j. (See Proposition 1.) It only remains to show that $\|t_j\|_\infty \to 0$ as $j \to +\infty$.

Routine calculations and the properties of $f^\#$ show that

$$|t_j(g)| \leqslant \left\{\lambda(g)\,\mu\left(E_j \cap \left(g^{-1}E_0\right)\right)\right\}^{1/2}.$$

If μ is G-invariant, $\lambda(g) = 1$ for every $g \in G$ and $\mu(E_j \cap (g^{-1}E_0))$ is dominated by $\mu(E_j)$ uniformly with respect to $g \in G$, hence $\|t_j\|_\infty \to 0$. Otherwise it is easy to see that

$$Z = \left\{ g\colon E_j \cap \left(g^{-1}E_0\right) \neq \varnothing \right\} = C_j N C_0^{-1},$$

where C_j and C_0 are compact sets in G such that $p(C_j) = E_j$. We have $\lambda(g) \leqslant$ constant for $g \in C_j N C_0^{-1} \subset C_1 N C_0^{-1}$, since N is compact and λ continuous, being the modular function of a relatively invariant measure, and this completes the proof.

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