HUREWICZ ISOMORPHISM THEOREM FOR STEENROD HOMOLOGY

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ABSTRACT. For a pointed compactum \((X, x)\), a natural homomorphism \(\xi_n\) from the Quigley’s approaching group \(\pi_n(X, x)\) to the Steenrod homology group \(H_{n+1}(X)\) is defined. A shape theoretical condition under which \(\xi_n\) is an isomorphism is obtained. For every pointed \(S^n\)-like continuum \((X, x)\), \(\xi_n\) is an isomorphism for \(n \neq 2\) and \(\xi_2\) is an isomorphism if and only if \(X\) is movable.

1. Introduction. Let \(X\) be a compact metric space and let \(x \in X\). Denote by \(H_{n}(X)\) the \(n\)-dimensional Steenrod homology group of \(X\) defined by Steenrod [14] and by \(\pi_n(X, x)\) the \(n\)-dimensional approaching group of \((X, x)\) defined by Quigley [13]. For each \(n\), there exists a natural homomorphism \(\xi_n: \pi_n(X, x) \rightarrow H_{n+1}(X)\). The purpose of this note is to establish a shape theoretical condition for \(X\) under which \(\xi_n\) is an isomorphism.

Throughout the paper all spaces are metrizable and maps are continuous. We denote by \(J\) the directed set of nonnegative integers.

2. \((m, n)\)-movability. Let \((K, v)\) be a pointed polyhedron and let \((Y, y)\) be a pointed space. For \(k \in J\), a map \(f: (Y, y) \rightarrow (K, v)\) is said to be \(k\)-deformable if there exists a map \(g: (Y, y) \rightarrow (K, v)\) such that \(f \simeq g \text{ rel } \{y\}\) and \(g(Y)\) is contained in a combinatorial \(k\)-skeleton of a triangulation of \(K\). For a pair \((n, r)\), \(n, r \in J\), a pointed compactum \((X, x)\) is said to be \((m, n)\)-movable if there exists an inverse sequence \(\{(X_i, x_i), \pi_i\}\) consisting of pointed finite polyhedra \((X_i, x_i), i \in J\), and bonding maps \(\pi_j: (X_j, x_j) \rightarrow (X_i, x_i), i < j, i, j \in J\), satisfying the following conditions:

(2.1) \((X, x) = \lim_{\leftarrow} (X_i, x_i)\).

(2.2) For every \(i \in J\), there exists \(j \in J, j > i\), such that for every \(k \in J, k > i\), and every \(n\)-deformable map \(f: (S^m, s) \rightarrow (X_j, x_j)\), where \((S^m, s)\) is a pointed \(m\)-sphere, there exists an \(n\)-deformable map \(g: (S^m, s) \rightarrow (X_k, x_k)\) such that \(\pi_kf \simeq \pi_kg \text{ rel } s\).

For a pointed compactum \((X, x)\), denote by \(\text{Sh}(X, x)\) the shape of \((X, x)\) defined by Borsuk [2]. The \((m, n)\)-movability of \((X, x)\) is a hereditary shape invariant in the sense of Borsuk, that is, we have
Theorem 1. Let \((X, x)\) and \((Y, y)\) be pointed compacta. If \(\text{Sh}(X, x) > \text{Sh}(Y, y)\) and \((X, x)\) is \((m, n)\)-movable, then \((Y, y)\) is \((m, n)\)-movable.

The proof is given by a standard technique (cf. [7, Theorem 1]) and we omit it.

For \(n \in J\), the pointed \(n\)-movability of \((X, x)\) is defined in a similar way to one given by Borsuk [1], that only the category of pointed compacta is considered. The following lemma is obvious from [7].

Lemma 1. A pointed \(n\)-movable compactum is \((m, n)\)-movable for every \(m \in J\).

Since the pointed movability (cf. [2, p. 166]) implies the pointed \(n\)-movability for every \(n \in J\), we have

Corollary 1. A pointed movable compactum is \((m, n)\)-movable for \(m, n \in J\).

Example 1. Let \(n \in J\), \(n > 0\). Let \(\{X_i, p_{i+1}\}\) be an inverse sequence consisting of \(n\)-spheres \(X_i\), \(i \in J\), such that each bonding map \(p_{i+1}: X_{i+1} \to X_i\) is of degree 2. Let \(X(n)\) be the limit space of \(\{X_i\}\) and \(x \in X(n)\). Then we have

(2.3) if \(n \neq 2\), \((X(n), x)\) is \((n + 1, n)\)-movable,

(2.4) \((X(2), x)\) is not \((3, 2)\)-movable.

To see (2.3) it is enough to note that \(\pi_2(S^1) = 0\) and \(\pi_{n+1}(S^n) = \mathbb{Z}_2\) for \(n > 3\), where \(\pi_i(Y)\) is the \(i\)-homotopy group of \(Y\). (Note that \(X(1)\) is \((k, 1)\)-movable for each \(k > 1\).)

Assertion (2.4) follows from \(\pi_3(S^2) = \mathbb{Z}\) (see [6, Chapter VI, Lemma 1.2]) and the definition of the \((3, 2)\)-movability.

Since \((X(n), x)\) is not pointed \(n\)-movable, the converse assertion of Lemma 1 or Corollary 1 does not generally hold.

3. Hurewicz isomorphism theorem. For a compactum \(X\), let \(\overset{\cdot}{H}_n(X)\) be the homology group of the regular \(n\)-cycles of \(X\) defined by N. E. Steenrod [14]. A beautiful description of \(\overset{\cdot}{H}_n(X)\) was given by J. Milnor [11]. For a pointed compactum \((X, x)\), Quigley [13] defined the approaching group \(\pi_n^+(X, x)\). To define a natural homomorphism \(\xi_n: \pi_n^+(X, x) \to \overset{\cdot}{H}_{n+1}(X)\), consider \(X\) as a subset of the Hilbert cube \(Q\). For an element \(\alpha \in \pi_n^+(X, x)\), let \(f: R^+ \times (S^n, s) \to (Q, x)\) be an approaching \(n\)-mapping representing in the sense of Quigley [13], where \(R^+ = \{t: 0 < t < \infty\}\). Let \(D^{n+1}\) be an \((n + 1)\)-ball whose boundary is \(S^n\) and put \(K = \{0\} \times D^{n+1} \cup R^+ \times S^n\). Let \(g: K \to Q\) be an extension of \(f\). Since \(K\) is an infinite \((n + 1)\)-cycle, the triple \((K, f, K)\) is an infinite \((n + 1)\)-cycle regular to \(X\) in the sense of Steenrod [14, p. 837]. Let \(\beta\) be the element of \(\overset{\cdot}{H}_{n+1}(X)\) represented by \((K, f, K)\). Obviously \(\beta\) is uniquely determined by the element \(\alpha\). Define \(\xi_n: \pi_n^+(X, x) \to \overset{\cdot}{H}_{n+1}(X)\) by \(\xi_n(\alpha) = \beta\). It is easy to show that \(\xi_n\) is a natural homomorphism.
Theorem 2. Let \((X, x)\) be a pointed compactum and let \(n \in J, n > 1\). If 
\((X, x)\) is \((n + 1, n)\)-movable and approximately \(k\)-connected for \(k = 0, 1, \ldots, n - 1\) [2, p. 145], then \(\xi_n: \pi_n(X, x) \rightarrow H_{n+1}(X)\) is an isomorphism.

Proof. Let \(\{(X_i, x_i), p_{i,i+1}\}\) be an inverse sequence consisting of pointed polyhedra such that \((X, x) = \lim \{(X_i, x_i), p_{i,i+1}\}\). Let \(\{H_k(X_i)\}\) be the inverse sequence consisting of the \(k\)-homology groups. Similarly let \(\{\pi_k(X_i, x_i)\}\) be the inverse sequence consisting of \(k\)-homotopy groups of \((X_i, x_i), i \in J\). From the proofs of [14, Theorem 7], [11, Theorem 4] and [15, Theorem 2] it is seen that the homomorphism \(\xi_n\) induces homomorphisms \(\mu\) and \(\eta_n\) such that the following diagram is commutative,

\[
\begin{array}{c}
0 \rightarrow \lim_{\leftarrow} \{(\pi_{n+1}(X_i, x_i)) \rightarrow \pi_n(X, x) \rightarrow \pi_{n}(X, x) \rightarrow 0
\end{array}

\downarrow \mu \hspace{1cm} \downarrow \xi_n \hspace{1cm} \downarrow \eta_n

\begin{array}{c}
0 \rightarrow \lim_{\leftarrow} \{H_{n+1}(X_i)\} \rightarrow \tilde{H}_{n+1}(X) \rightarrow \tilde{H}_{n}(X) \rightarrow 0
\end{array}

\tag{3.1}
\]

Here \(\tilde{H}_n(X)\) is the Čech \(n\)-homology group of \(X\), \(\pi_n(X, x)\) is the \(n\)th fundamental group defined by Borsuk [2, Chapter XII] and \(\lim(1)\) is the first derived functor of the inverse limit functor \(\lim\). The homomorphism \(\mu\) is induced by the Hurewicz homomorphism \(\mu_j: \pi_{n+1}(X_i, x_i) \rightarrow H_{n+1}(X_i), j \in J\), and \(\eta_n\) is the limit Hurewicz homomorphism in the sense of Kuperberg [9, p. 26]. The exactness of the top row of diagram (3.1) follows from Theorem 2 of Watanabe [15] (cf. Grossman [5] or Edwards and Hastings [4, 5.2.1]). Milnor [11, Theorem 4] proved the bottom row of (3.1) is exact. Since \((X, x)\) is approximatively \(k\)-connected for \(k = 0, 1, \ldots, n - 1\), by [9, Theorem 3.2] \(\eta_n\) is an isomorphism. It remains to prove that \(\mu\) is an isomorphism. To show it, note that we may assume every \((X_i, x_i), i \in J\), is \((n - 1)\)-connected. (This is proved by the same way as in Lemma (1.6) of Nowak [12]). Then the Hurewicz homomorphism \(\mu_j: \pi_{n+1}(X_i, x_i) \rightarrow H_{n+1}(X_i)\) is onto by [6, Theorem 2.6]. Let \(G_i = \text{Kernel } \mu_i, i \in J\). Then \(\{G_i\}\) forms an inverse sequence. Consider the following exact sequence in the category \(\text{pro-}G\) of pro-groups:

\[0 \rightarrow \{G_i\} \rightarrow \{\pi_{n+1}(X_i, x_i)\} \rightarrow \{H_{n+1}(X_i)\} \rightarrow 0, \hspace{1cm} \tag{3.2}\]

where \(j_i: G_i \rightarrow \pi_{n+1}(X_i, x_i)\) is the inclusion homomorphism, \(i \in J\), and 0 means a zero object in \(\text{pro-}G\). By [3, p. 256] the sequence (3.2) induces the exact sequence:

\[
\lim_{\leftarrow} \{(j_i)\} \rightarrow \lim_{\leftarrow} \{\pi_{n+1}(X_i, x_i)\} \rightarrow \lim_{\leftarrow} \{H_{n+1}(X_i)\} \rightarrow 0. \hspace{1cm} \tag{3.3}\]

Let \(f: (S^{n+1}, s) \rightarrow (X_i, x_i)\) be a map representing an element \(\alpha \in \pi_{n+1}(X_i, x_i)\). Then \(\alpha \in G_i\) if and only if \(f\) is \(n\)-deformable. From this fact and the \((n + 1, n)\)-movability of \((X, x)\) it follows that the inverse sequence \(\{G_i\}\) satisfies the Mittag-Leffler condition. Thus \(\lim_{\leftarrow} \{G_i\} = 0\) by [3, p. 256]. The
exactness of the sequence (3.3) shows that $\mu$ is an isomorphism. This completes the proof.

**Corollary 2.** Let $(X, x)$ be a pointed movable compactum such that $\pi_i(X, x) = 0$ for $i = 0, 1, \ldots, n - 1$, $n > 1$. Then $\pi_n(X, x) = \pi_n(X, x) = H_n(X) = H_{n+1}(X)$.

This is obvious from Corollary 1, Theorem 2 and Borsuk [2, Chapter V, Theorem (10.1)].

**Corollary 3.** Let $(X, x)$ be approximatively connected for $k = 0, 1, \ldots, n - 1$. If there exists an inverse sequence $\{(X_i, x_i)\}$ such that $(X, x) = \lim (X_i, x_i)$ and $\lim \pi_n(X_i, x_i) = 0$, then $\xi_n : \pi_n(X, x) \to H_{n+1}(X)$ is an isomorphism.

**Proof.** By [8], $(X, x)$ is pointed $S^n$-movable. Since the approximative $k$-connectedness for $k = 0, 1, \ldots, n - 1$ and the pointed $S^n$-movability imply the pointed $n$-movability, the corollary follows from Lemma 1 and Theorem 2.

**Corollary 4.** Let $X$ be an $S^n$-like continuum and let $n \neq 2$. Then, for every point $x \in X$, the homomorphism $\xi_n : \pi_n(X, x) \to H_{n+1}(X)$ is an isomorphism.

**Proof.** By (2.3) of Example 1, $(X, x)$ is $(n + 1, n)$-movable. Thus, if $n > 2$ the corollary follows from Theorem 2. For $n = 1$, consider the diagram (3.1). Obviously $\lim \pi_2(X_i, x_i) = \lim \pi_2(H_2(X_i)) = 0$ and $\eta_1$ is an isomorphism. Thus $\xi_1$ is an isomorphism.

**Corollary 5.** Let $X$ be an $S^2$-like continuum and let $x \in X$. Then the following conditions are equivalent:

(3.4) $(X, x)$ is pointed movable.

(3.5) $\xi_2 : \pi_2(X, x) \to H_3(X)$ is an isomorphism.

**Proof.** The implication $(3.4) \to (3.5)$ is a consequence of Theorem 2. To prove $(3.5) \to (3.4)$, assume that $(X, x)$ is not pointed movable. Let $\{(X_i, x_i), p_{i, i+1}\}$ be an inverse sequence consisting of 2-spheres such that $(X, x) = \lim (X_i, x_i)$. By Mardešić and Segal [10, Theorem 4], $X$ is neither of trivial shape nor of the shape of $S^2$. Then, for an infinite number of $i \in J$, the bonding map $p_{i, i+1}$ has degree $> 2$. Since $\pi_3(X_i, x_i) = \mathbb{Z}$, we have $\lim \pi_3(X_i, x_i) \neq 0$. On the other hand, $\lim H_3(X_i) = 0$. Thus the homomorphism $\mu$ in (3.1) is not an isomorphism. Hence $\xi_2$ is not an isomorphism.

Example 1 and Corollary 5 imply that we cannot omit the $(n + 1, n)$-movability or the approximative connectivity of $(X, x)$ in Theorem 2. Finally, we give an example of a pointed continuum $(X, x)$ such that $(X, x)$ is approximatively 1-connected but not pointed $(3, 2)$-movable, and $\xi_2$ is an isomorphism.

**Example 2.** For $n \in J$, let $X_n$ be a one point union of a 2-sphere $S_n$ and
3-spheres $S_{n,i}, i = 1, 2, \ldots, n$, with the base point $x_n$. Let $p_{n,n+1}: (X_{n+1}, x_{n+1}) \to (X_n, x_n)$ be a map such that

- $p_{n,n+1}|_{S_{n+1}}$ is a map from $S_{n+1}$ to $S_n$ with degree $2$,
- $p_{n,n+1}|_{S_{n+1,i}}$ is the Hopf map from $S_{n+1,i}$ to $S_n$,
- $p_{n,n+1}|_{S_{n+1,i}, i = 2, \ldots, n+1}$ is a homeomorphism from $S_{n+1,i}$ to $S_{n,i-1}$.

Let $(X, x)$ be the limit of the inverse sequence $\{ (X_n, x_n), p_{n,n+1} \}$. As shown in Example 1, $(X, x)$ is not $(3, 2)$-movable. Obviously $\pi_3(X, x) = \tilde{H}_3(X) = 0$.

Since both of the inverse sequences $\{ \pi_3(X_n, x_n) \}$ and $\{ H_3(X_n) \}$ satisfy the Mittag-Leffler condition, by diagram (3.1) we have $\pi_3(X, x) = \tilde{H}_3(X) = 0$.

**REFERENCES**


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