PROCEEDINGS OF THE
AMERICAN MATHEMATICAL SOCIETY
Volume 74, Number 2, May 1979

HUREWICZ ISOMORPHISM THEOREM FOR STEENROD HOMOLOGY

Y. KODAMA AND A. KOYAMA

ABSTRACT. For a pointed compactum (X, x), a natural homomorphism ξ_n from the Quigley's approaching group $\underline{\pi}_n(X, x)$ to the Steenrod homology group ${}^sH_{n+1}(X)$ is defined. A shape theoretical condition under which ξ_n is an isomorphism is obtained. For every pointed S^n-like continuum (X, x), ξ_n is an isomorphism for $n \neq 2$ and ξ_2 is an isomorphism if and only if X is movable.

1. Introduction. Let X be a compact metric space and let $x \in X$. Denote by ${}^sH_n(X)$ the n-dimensional Steenrod homology group of X defined by Steenrod [14] and by $\underline{\pi}_n(X, x)$ the n-dimensional approaching group of (X, x) defined by Quigley [13]. For each n, there exists a natural homomorphism ξ_n: $\underline{\pi}_n(X, x) \to {}^sH_{n+1}(X)$. The purpose of this note is to establish a shape theoretical condition for X under which ξ_n is an isomorphism.

Throughout the paper all spaces are metrizable and maps are continuous. We denote by J the directed set of nonnegative integers.

2. (m, n)-movability. Let (K, v) be a pointed polyhedron and let (Y, y) be a pointed space. For $k \in J$, a map f: $(Y, y) \to (K, v)$ is said to be *k-deformable* if there exists a map g: $(Y, y) \to (K, v)$ such that $f \simeq g$ rel $\{y\}$ and $g(Y)$ is contained in a combinatorial k-skeleton of a triangulation of K. For a pair (n, r), $n, r \in J$, a pointed compactum (X, x) is said to be *(m, n)-movable* if there exists an inverse sequence $\{(X_i, x_i), \pi_{ij}\}$ consisting of pointed finite polyhedra (X_i, x_i), $i \in J$, and bonding maps π_{ij}: $(X_j, x_j) \to (X_i, x_i)$, $i < j$, $i, j \in J$, satisfying the following conditions:

(2.1) $(X, x) = \varprojlim (X_i, x_i)$.

(2.2) For every $i \in J$, there exists $j \in J$, $j \geqslant i$, such that for every $k \in J$, $k \geqslant i$, and every n-deformable map f: $(S^m, s) \to (X_j, x_j)$, where (S^m, s) is a pointed m-sphere, there exists an n-deformable map g: $(S^m, s) \to (X_k, x_k)$ such that $\pi_{ij} f \simeq \pi_{ik} g$ rel s.

For a pointed compactum (X, x), denote by $\mathrm{Sh}(X, x)$ the shape of (X, x) defined by Borsuk [2]. The (m, n)-movability of (X, x) is a hereditary shape invariant in the sense of Borsuk, that is, we have

Received by the editors February 15, 1978 and, in revised form, June 13, 1978.

AMS (MOS) subject classifications (1970). Primary 54F99; Secondary 55B99, 55E99.

Key words and phrases. Shape, movability, Steenrod homology, Hurewicz isomorphism theorem.

THEOREM 1. *Let* (X, x) *and* (Y, y) *be pointed compacta. If* $\mathrm{Sh}(X, x) \geqslant \mathrm{Sh}(Y, y)$ *and* (X, x) *is* (m, n)*-movable, then* (Y, y) *is* (m, n)*-movable.*

The proof is given by a standard technique (cf. [**7**, Theorem 1]) and we omit it.

For $n \in J$, the pointed n-movability of (X, x) is defined in a similar way to one given by Borsuk [**1**], that only the category of pointed compacta is considered. The following lemma is obvious from [**7**].

LEMMA 1. *A pointed* n*-movable compactum is* (m, n)*-movable for every* $m \in J$.

Since the pointed movability (cf. [**2**, p. 166]) implies the pointed n-movability for every $n \in J$, we have

COROLLARY 1. *A pointed movable compactum is* (m, n)*-movable for* m, $n \in J$.

EXAMPLE 1. Let $n \in J$, $n > 0$. Let $\{X_i, p_{i,i+1}\}$ be an inverse sequence consisting of n-spheres X_i, $i \in J$, such that each bonding map $p_{i,i+1}\colon X_{i+1} \to X_i$ is of degree 2. Let $X(n)$ be the limit space of $\{X_i\}$ and $x \in X(n)$. Then we have

(2.3) if $n \neq 2$, $(X(n), x)$ is $(n + 1, n)$-movable,

(2.4) $(X(2), x)$ is not $(3, 2)$-movable.

To see (2.3) it is enough to note that $\pi_2(S^1) = 0$ and $\pi_{n+1}(S^n) = Z_2$ for $n \geqslant 3$, where $\pi_i(Y)$ is the i-homotopy group of Y. (Note that $X(1)$ is $(k, 1)$-movable for each $k > 1$.)

Assertion (2.4) follows from $\pi_3(S^2) = Z$ (see [**6**, Chapter VI, Lemma 1.2]) and the definition of the $(3, 2)$-movability.

Since $(X(n), x)$ is not pointed n-movable, the converse assertion of Lemma 1 or Corollary 1 does not generally hold.

3. Hurewicz isomorphism theorem. For a compactum X, let ${}^sH_n(X)$ be the homology group of the regular n-cycles of X defined by N. E. Steenrod [**14**]. A beautiful description of ${}^sH_n(X)$ was given by J. Milnor [**11**]. For a pointed compactum (X, x), Quigley [**13**] defined the approaching group $\underline{\underline{\pi}}_n(X, x)$. To define a natural homomorphism $\xi_n\colon \underline{\underline{\pi}}_n(X, x) \to {}^sH_{n+1}(X)$, consider X as a subset of the Hilbert cube Q. For an element $\alpha \in \underline{\underline{\pi}}_n(X, x)$, let $f\colon R^+ \times (S^n, s) \to (Q, x)$ be an approaching n-mapping representing in the sense of Quigley [**13**], where $R^+ = \{t\colon 0 \leqslant t < \infty\}$. Let D^{n+1} be an $(n + 1)$-ball whose boundary is S^n and put $K = \{0\} \times D^{n+1} \cup R^+ \times S^n$. Let $g\colon K \to Q$ be an extension of f. Since K is an infinite $(n + 1)$-cycle, the triple (K, f, K) is an infinite $(n + 1)$-cycle regular to X in the sense of Steenrod [**14**, p. 837]. Let β be the element of ${}^sH_{n+1}(X)$ represented by (K, f, K). Obviously β is uniquely determined by the element α. Define $\xi_n\colon \underline{\underline{\pi}}_n(X, x) \to {}^sH_{n+1}(X)$ by $\xi_n(\alpha) = \beta$. It is easy to show that ξ_n is a natural homomorphism.

THEOREM 2. *Let (X, x) be a pointed compactum and let $n \in J$, $n > 1$. If (X, x) is $(n+1, n)$-movable and approximatively k-connected for $k = 0, 1, \ldots, n-1$* [**2**, *p.* 145], *then ξ_n: $\underline{\pi}_n(X, x) \to {}^s H_{n+1}(X)$ is an isomorphism.*

PROOF. Let $\{(X_i, x_i), p_{i,i+1}\}$ be an inverse sequence consisting of pointed polyhedra such that $(X, x) = \varprojlim \{(X_i, x_i), p_{i,i+1}\}$. Let $\{H_k(X_i)\}$ be the inverse sequence consisting of the k-homology groups. Similarly let $\{\pi_k(X_i, x_i)\}$ be the inverse sequence consisting of k-homotopy groups of (X_i, x_i), $i \in J$. From the proofs of [**14**, Theorem 7], [**11**, Theorem 4] and [**15**, Theorem 2] it is seen that the homomorphism ξ_n induces homomorphisms μ and η_n such that the following diagram is commutative,

$$\begin{array}{ccccccccc} 0 & \to & \varprojlim^{(1)}\{\pi_{n+1}(X_i, x_i)\} & \to & \underline{\pi}_n(X, x) & \to & \pi_n(X, x) & \to & 0 \\ & & \downarrow \mu & & \downarrow \xi_n & & \downarrow \eta_n & & \\ 0 & \to & \varprojlim^{(1)}\{H_{n+1}(X_i)\} & \to & {}^s H_{n+1}(X) & \to & \check{H}_n(X) & \to & 0 \end{array} \tag{3.1}$$

Here $\check{H}_n(X)$ is the Čech n-homology group of X, $\underline{\pi}_n(X, x)$ is the nth fundamental group defined by Borsuk [**2**, Chapter XII] and $\varprojlim^{(1)}$ is the first derived functor of the inverse limit functor $\varprojlim$. The homomorphism μ is induced by the Hurewicz homomorphism μ_i: $\pi_{n+1}(X_i, x_i) \to H_{n+1}(X_i)$, $j \in J$, and η_n is the limit Hurewicz homomorphism in the sense of Kuperberg [**9**, p. 26]. The exactness of the top row of diagram (3.1) follows from Theorem 2 of Watanabe [**15**] (cf. Grossman [**5**] or Edwards and Hastings [**4**, 5.2.1]). Milnor [**11**, Theorem 4] proved the bottom row of (3.1) is exact. Since (X, x) is approximatively k-connected for $k = 0, 1, \ldots, n-1$, by [**9**, Theorem 3.2] η_n is an isomorphism. It remains to prove that μ is an isomorphism. To show it, note that we may assume every (X_i, x_i), $i \in J$, is $(n-1)$-connected. (This is proved by the same way as in Lemma (1.6) of Nowak [**12**].) Then the Hurewicz homomorphism μ_i: $\pi_{n+1}(X_i, x_i) \to H_{n+1}(X_i)$ is onto by [**6**, Theorem 2.6]. Let $G_i = \text{Kernel } \mu_i$, $i \in J$. Then $\{G_i\}$ forms an inverse sequence. Consider the following exact sequence in the category pro-G of pro-groups:

$$0 \to \{G_i\} \xrightarrow{\{j_i\}} \{\pi_{n+1}(X_i, x_i)\} \to \{H_{n+1}(X_i)\} \to 0, \tag{3.2}$$

where j_i: $G_i \to \pi_{n+1}(X_i, x_i)$ is the inclusion homomorphism, $i \in J$, and 0 means a zero object in pro-G. By [**3**, p. 256] the sequence (3.2) induces the exact sequence:

$$\varprojlim^{(1)}\{G_i\} \xrightarrow{\varprojlim\{j_i\}} \varprojlim^{(1)}\{\pi_{n+1}(X_i, x_i)\} \to \varprojlim^{(1)}\{H_{n+1}(X_i)\} \to 0. \tag{3.3}$$

Let f: $(S^{n+1}, s) \to (X_i, x_i)$ be a map representing an element $\alpha \in \pi_{n+1}(X_i, x_i)$. Then $\alpha \in G_i$ if and only if f is n-deformable. From this fact and the $(n+1, n)$-movability of (X, x) it follows that the inverse sequence $\{G_i\}$ satisfies the Mittag-Leffler condition. Thus $\varprojlim^{(1)}\{G_i\} = 0$ by [**3**, p. 256]. The

exactness of the sequence (3.3) shows that μ is an isomorphism. This completes the proof.

COROLLARY 2. *Let (X, x) be a pointed movable compactum such that $\underline{\pi}_i(X, x) = 0$ for $i = 0, 1, \ldots, n-1$, $n > 1$. Then $\underline{\pi}_n(X, x) = \underline{\underline{\pi}}_n(X, x) = \check{H}_n(X) = {}^sH_{n+1}(X)$.*

This is obvious from Corollary 1, Theorem 2 and Borsuk [**2**, Chapter V, Theorem (10.1)].

COROLLARY 3. *Let (X, x) be approximatively connected for $k = 0, 1, \ldots, n-1$. If there exists an inverse sequence $\{(X_i, x_i)\}$ such that $(X, x) = \varprojlim(X_i, x_i)$ and $\varprojlim^{(1)}\{\pi_n(X_i, x_i)\} = 0$, then ξ_n: $\underline{\underline{\pi}}_n(X, x) \to {}^sH_{n+1}(X)$ is an isomorphism.*

PROOF. By [**8**], (X, x) is pointed S^n-movable. Since the approximative k-connectedness for $k = 0, 1, \ldots, n-1$ and the pointed S^n-movability imply the pointed n-movability, the corollary follows from Lemma 1 and Theorem 2.

COROLLARY 4. *Let X be an S^n-like continuum and let $n \neq 2$. Then, for every point $x \in X$, the homomorphism ξ_n: $\underline{\underline{\pi}}_n(X, x) \to {}^sH_{n+1}(X)$ is an isomorphism.*

PROOF. By (2.3) of Example 1, (X, x) is $(n+1, n)$-movable. Thus, if $n > 2$ the corollary follows from Theorem 2. For $n = 1$, consider the diagram (3.1). Obviously $\varprojlim^{(1)}\{\pi_2(X_i, x_i)\} = \varprojlim^{(1)}\{H_2(X_i)\} = 0$ and η_1 is an isomorphism. Thus ξ_1 is an isomorphism.

COROLLARY 5. *Let X be an S^2-like continuum and let $x \in X$. Then the following conditions are equivalent*:

(3.4) *(X, x) is pointed movable.*

(3.5) *ξ_2: $\underline{\underline{\pi}}_2(X, x) \to {}^sH_3(X)$ is an isomorphism.*

PROOF. The implication (3.4) $\to$ (3.5) is a consequence of Theorem 2. To prove (3.5) $\to$ (3.4), assume that (X, x) is not pointed movable. Let $\{(X_i, x_i), p_{i,i+1}\}$ be an inverse sequence consisting of 2-spheres such that $(X, x) = \varprojlim(X_i, x_i)$. By Mardešić and Segal [**10**, Theorem 4], X is neither of trivial shape nor of the shape of S^2. Then, for an infinite number of $i \in J$, the bonding map $p_{i,i+1}$ has degree $\geqslant 2$. Since $\pi_3(X_i, x_i) = Z$, we have $\varprojlim^{(1)}\{\pi_3(X_i, x_i)\} \neq 0$. On the other hand, $\varprojlim^{(1)}\{H_3(X_i)\} = 0$. Thus the homomorphism μ in (3.1) is not an isomorphism. Hence ξ_2 is not an isomorphism.

Example 1 and Corollary 5 imply that we cannot omit the $(n+1, n)$-movability or the approximative connectivity of (X, x) in Theorem 2. Finally, we give an example of a pointed continuum (X, x) such that (X, x) is approximatively 1-connected but not pointed (3, 2)-movable, and ξ_2 is an isomorphism.

EXAMPLE 2. For $n \in J$, let X_n be a one point union of a 2-sphere S_n and

3-spheres $S_{n,i}$, $i = 1, 2, \ldots, n$, with the base point x_n. Let $p_{n,n+1}\colon (X_{n+1}, x_{n+1}) \to (X_n, x_n)$ be a map such that

$p_{n,n+1}|S_{n+1}$ is a map from S_{n+1} to S_n with degree 2,

$p_{n,n+1}|S_{n+1,1}$ is the Hopf map from $S_{n+1,1}$ to S_n,

$p_{n,n+1}|S_{n+1,i}$, $i = 2, \ldots, n+1$, is a homeomorphism from $S_{n+1,i}$ to $S_{n,i-1}$.

Let (X, x) be the limit of the inverse sequence $\{(X_n, x_n), p_{n,n+1}\}$. As shown in Example 1, (X, x) is not (3, 2)-movable. Obviously $\underline{\pi}_2(X, x) = \check{H}_2(X) = 0$. Since both of the inverse sequences $\{\pi_3(X_n, x_n)\}$ and $\{H_3(X_n)\}$ satisfy the Mittag-Leffler condition, by diagram (3.1) we have $\underline{\underline{\pi}}_2(X, x) = {}^sH_3(X) = 0$.

References

[1] K. Borsuk, *On the n-movability*, Bull. Acad. Polon. Sci. Sér. Sci. Math. Astronom. Phys. **20** (1972), 859–864.

[2] _____, *Theory of shape*, PWN–Polish Scientific Publishers, Warsaw, 1975.

[3] A. K. Bousfield and D. M. Kan, *Homotopy limits, completions and localizations*, Lecture Notes in Math., vol. 304, Springer-Verlag, Berlin and New York, 1972.

[4] D. A. Edwards and H. M. Hastings, *Čech and Steenrod homotopy theories with applications to geometric topology*, Lecture Notes in Math., vol. 542, Springer-Verlag, Berlin and New York, 1976.

[5] J. Grossman, *Homotopy classes of maps between pro-spaces*, Michigan Math. J. **21** (1974), 355–362.

[6] P. J. Hilton, *An introduction to homotopy theory*, Cambridge Univ. Press, London and New York, 1953.

[7] Y. Kodama and T. Watanabe, *A note on Borsuk's n-movability*, Bull. Acad. Polon. Sci. Sér. Sci. Math. Astronom. Phys. **22** (1974), 289–294.

[8] A. Koyama, J. Ono and K. Tsuda, *An algebraic characterization of pointed S^n-movability*, Bull. Acad. Polon. Sci. Sér. Sci. Math. Astronom. Phys. **25** (1977), 1249–1252.

[9] K. Kuperberg, *An isomorphism theorem of the Hurewicz type in Borsuk's theory of shape*, Fund. Math. **77** (1972), 21–32.

[10] S. Mardešić and J. Segal, *Movable compacta and ANR-systems*, Bull. Acad. Polon. Sci. Sér. Sci. Math. Astronom. Phys. **18** (1970), 649–654.

[11] J. Milnor, *On the Steenrod homology*, Mimeographed notes, Berkeley, 1960.

[12] S. Nowak, *On the fundamental dimension of approximatively* 1*-connected compacta*, Fund. Math. **89** (1975), 61–79.

[13] J. B. Quigley, *An exact sequence from the nth to the* $(n-1)$*-st fundamental group*, Fund. Math. **77** (1973), 195–210.

[14] N. E. Steenrod, *Regular cycles of compact metric spaces*, Ann. of Math. **41** (1940), 833–851.

[15] T. Watanabe, *On a problem of Y. Kodama*, Bull. Acad. Polon. Sci. Sér. Sci. Math. Astronom. Phys. **25** (1977), 981–986.

Department of Mathematics, University of Tsukuba, Ibaraki, Japan