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A CHARACTERIZATION OF COMPACT GROUPS

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ABSTRACT. It is shown that the group algebra $L^1(G)$ of a locally compact group G is an ideal in the bidual $L^1(G)^{**}$ of $L^1(G)$ (equipped with Arens product) if and only if G is compact.

Let G be a locally compact group with group algebra $L^1(G)$ and let the bidual $L^1(G)^{**}$ of $L^1(G)$ be equipped with an Arens product as in [**2**]. It is known [**5**, Corollary 3.4] that, if G is compact, then $L^1(G)$ is an ideal in $L^1(G)^{**}$. The converse was asserted in [**4**, Theorem 7.5]; however, the "proof" given there was based upon a false result (Lemma 7.1 of [**4**]). In this note, we give a proof of the converse and also show why [**4**, Lemma 7.1] is false in the generality stated.

THEOREM 1. *A locally compact group G is compact if and only if $L^1(G)$ is an ideal in $L^1(G)^{**}$.*

PROOF. We will only establish the implication ($\Leftarrow$). Let $W(G) \subset L^\infty(G)$ be the commutative C^*-algebra of all continuous weakly almost periodic functions on G (see [**1**]), and let G_W be its maximal ideal space. Then

$$M(G_W) = W(G)^* = L^1(G)^{**}/W(G)^\perp$$

and, because $L^1(G)$ is an ideal in $L^1(G)^{**}$, it is also an ideal in $M(G_W)$. Thus, by [3, Theorem 2.1, Corollary 2.3], $M(G_W) = M(G)$ and, as a result, $C_0(G) = W(G)$. However, the function 1 is in $W(G)$, so G is compact. □

The proof given above was constructed jointly by the author and Charles D. Lahr; it replaces a more elementary, but slightly longer, proof due to the author.

Now, let A be a Banach algebra, let $M_r(A)$ be the Banach algebra of all continuous right multipliers of A and let $G_r(A)$ be the subset of $M_r(A)$ consisting of all isometric onto right multipliers of A. Lemma 7.1 of [**4**] states that, if the closed unit ball of $M_r(A)$ is compact in the weak operator topology τ_r it inherits from $\mathbf{B}(A)$, then $(G_r(A), \tau_r)$ is a compact topological group. The following theorem shows that this is false even for dual C^*-algebras A (compare with [**4**, Theorem 7.4]). For such an A, it is known [**5**, Theorem 2.2] that $M_r(A) = A^{**}$, from which it follows easily that the unit ball of $M_r(A)$ is τ_r-compact.

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THEOREM 2. *Let $A = CC(H)$ be the dual C^*-algebra of all compact operators on a Hilbert space H. Then $G_r(A)$ is τ_r-compact (if and) only if H is finite dimensional.*

PROOF. First, the bidual A^{**} of A can be identified with $\mathbf{B}(H)$; thus, $G_r(A) \subset \mathbf{B}(H)$. In fact, it is routine to establish that $G_r(A)$ is the group $U(H)$ of unitary operators on H. Next, let $x, y \in H$ and let $T_\alpha \to T$ in the τ_r-topology on $G_r(A)$. Let M in A be such that $Mx = x$ (if $x \neq 0$, then $Mz = ((z|x)/\|x\|^2)x$ will do), and let M^* in A^* be defined by $\langle N, M^* \rangle = (Nx|y)$. Then

$$(T_\alpha x|y) = (T_\alpha Mx|y) = \langle T_\alpha M, M^* \rangle \to \langle TM, M^* \rangle = (TMx|y) = (Tx|y),$$

so $T_\alpha \to T$ in the weak operator topology τ_{wo} on $\mathbf{B}(H)$. Therefore, if $G_r(A)$ is τ_r-compact, then $U(H)$ is τ_{wo}-compact, and this cannot occur if H is infinite-dimensional. □

ADDED IN PROOF. It has recently come to my attention that Theorem 1 was proved, using different methods, by S. Watanabe [*A Banach algebra which is an ideal in the second dual space*, Sci. Rep. Niigata Univ. **11** (1974), 95–101].

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