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THE JACOBSON RADICAL OF THE GROUP ALGEBRA OF A FINITE GROUP

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ABSTRACT. Let K be a field of characteristic $p \neq 0$ and G a finite group such that $p \mid o(G)$. Suppose G is a Frobenius group with a Sylow p-subgroup P as a complement. Then we have proved that

$$JK(G) = \bigcap_{x \in G} JK(P^x)K(G).$$

We have given an example to show that equality does not hold in general.

0. Introduction. Let G be a finite group and K a field of characteristic $p \neq 0$. If $p \nmid o(G)$ then $JK(G) = 0$ and if $p | o(G)$ then $JK(G) \neq 0$ [**2**, Theorem 1.4.1]. If G is a p-group then [**2**, Theorem 2.3.2] $JK(G) = w(K(G))$, the augmentation ideal of $K(G)$. More generally [**5**, Theorem 16.6], if G is a group having unique Sylow p-subgroup P, then

$$JK(G) = w(K(P))K(G).$$

So one asks the following question:

Is $JK(G) = \bigcap JK(P)K(G)$ where P ranges over Sylow p-subgroups of G? We prove (Theorem 1 and Corollary 1) this to be the case for a group G having a normal subgroup G_0 such that $p \nmid (G : G_0)$ and that G_0 is a Frobenius group having a Sylow p-subgroup P as its complement subgroup. An example due to Passman shows that this equality is false in general.

1. Definitions and preliminaries. A finite group G is said to be a Frobenius group with complement H if H is a subgroup of G such that (1) $\{e\} \subsetneq H \subsetneq G$ and (2) $xHx^{-1} \cap H = (e)$ for every $x \in G - H$. A finite group G is a Frobenius group iff G is isomorphic to a transitive permutation group such that subgroup fixing any of the letters is nontrivial and each permutation $\neq e$ fixes at most one letter [**1**, Chapter 2, p. 37], [**4**, p. 57]. The most important fact regarding these groups is the following:

$$N = \{x \in G | C(x) \cap H = \varnothing\} \cup \{e\}$$
$$= \left(G - \bigcap_{x \in G} xHx^{-1}\right) \cup \{e\}$$

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is a normal subgroup of order $(G : H)$ and $G = NH$ [**1**, Theorem 7.5]. The subgroup N is called the Frobenius kernel of G. If $H = P$, Sylow p-subgroup of G then $p \nmid o(N)$ because $o(N) = (G : P)$. The number of distinct Sylow p-subgroups of G equals $o(N)$.

We say $x \in G$ is a p-element provided $o(x) = p^i$ for some $i \geqslant 0$. The p-trace of $\Sigma a_x x \in K(G)$ is defined to be

$$\mathrm{tr}_p(\Sigma a_x x) = \sum_{x \text{ is a } p\text{-element}} a_x.$$

LEMMA 1. *If $\alpha \in K(G)$ is nilpotent then* $\mathrm{tr}_p(\alpha) = 0$.

For proof see [**6**, Lemma 2.3.3].

LEMMA 2. *Suppose N and H are subgroups of the finite group G with $N \lhd G$ and $G = NH$. If $\hat{N} = \Sigma_{n \in N}\, n$, then*

$$JK(H)\hat{N} \subset \left(\bigcap_{x \in G} JK(H^x)K(G) \right) \cap JK(G).$$

PROOF. Observe that $\hat{N}$ is central in $K(G)$ and $n\hat{N} = \hat{N}$ for all $n \in N$. Let $x \in G = NH$ so $x = nh$, where $n \in N$ and $h \in H$. Then

$$\begin{aligned} JK(H^x)\hat{N} &= h^{-1}n^{-1}JK(H)nh\hat{N} \\ &= h^{-1}JK(H)h\hat{N} = JK(H)\hat{N}. \end{aligned}$$

Hence $JK(H)\hat{N} \subset \bigcap_{x \in G} JK(H^x)K(G)$. Since $(JK(H)\hat{N})x = x(JK(H)\hat{N})$ for all $x \in G$, $JK(H)\hat{N}$ generates a nilpotent two sided ideal of $K(G)$. Thus $JK(H)\hat{N} \subset JK(G)$.

2. Main result. From now on, we assume that K is an arbitrary field of characteristic $p \neq 0$, G a finite group such that $p \mid o(G)$, and P a fixed Sylow p-subgroup of G.

THEOREM 1. *Suppose G is a Frobenius group with Sylow p-subgroup P as a complement. Then $JK(G) = \bigcap_{x \in G} JK(P^x)K(G)$.*

PROOF. We shall prove this in two parts by showing that each side equals $JK(P)\hat{N}$ where N is the Frobenius kernel and $\hat{N}$ denotes the element $\Sigma_{x \in N} x$.

First we shall show that $JK(G) = JK(P)\hat{N}$. (Wallace [**7**] proved, using modular representation theory, that if G is a group such that $G'P$ is a Frobenius group with G' as the Frobenius kernel and P as a complement then $JK(G'P) = JK(P)\hat{G}'$. Our result is stronger and its proof uses simpler techniques.) By Lemma 2, $JK(P)\hat{N} \subset JK(G)$. Conversely, let $\alpha \in JK(G)$. Let

$$\alpha = \sum_{x \in N} a_x x + \sum_{y \in G - N} a_y y.$$

Since $JK(G)$ is nilpotent α is nilpotent. Thus by Lemma 1, we have $\mathrm{tr}_p(\alpha) = 0$. Since $p \nmid o(N)$ no element $x \neq 1$ in N is a p-element. If $y \in G - N$ then y

belongs to some Sylow p-subgroup of G. Thus y is a p-element. Therefore $\mathrm{tr}_p(\alpha) = 0$ gives

$$a_1 + \sum_{y \in G-N} a_y = 0.$$

Since $JK(G) \subset w(K(G))$ we have

$$\sum_{x \in N} a_x + \sum_{y \in G-N} a_y = 0$$

and hence $a_1 = \Sigma_{x \in N} a_x$. Now let $n \in N$. Then $\alpha n^{-1} \in JK(G)$ and

$$\alpha n^{-1} = \sum_{x \in N} a_x x n^{-1} + \sum_{y \in G-N} a_y y n^{-1}.$$

Therefore, by the same arguments, $a_n = \Sigma_{x \in N} a_x$ and hence $a_1 = a_n$ for all $n \in N$. Let $h \in P$. Then $\alpha h^{-1} \in JK(G)$ and $\alpha h^{-1} = (\Sigma_{x \in N} a_{xh} xh)h^{-1} + \alpha'$ where support$(\alpha') \subset G - N$. Then by the above argument, the coefficients of the elements of N in αh^{-1} are all equal. Therefore $a_h = a_{nh}$ for all $n \in N$. Hence $\alpha = \Sigma_{h \in P} a_h \hat{N} h$. Since $\alpha \in JK(G) \subset w(K(G))$ we have $(\Sigma_{h \in P} a_h) o(N) = 0$ and so $\Sigma_{h \in P} a_h = 0$ because $o(N) \neq 0$ in K. Thus $\Sigma_{h \in P} a_h h \in w(K(P)) = JK(P)$. Hence $\alpha \in JK(P)\hat{N}$.

Now we shall prove that $\Sigma_{x \in G} JK(P^x)K(G) = JK(P)\hat{N}$. By Lemma 2, we have

$$JK(P)\hat{N} \subset \bigcap_{x \in G} JK(P^x)K(G).$$

For the reverse inclusion, let $I = \bigcap_{x \in G} JK(P^x)K(G)$. Let $P_1 = P$, $P_2, \ldots, P_t$ be all the distinct Sylow p-subgroups of G. Then $t \neq 0$ in K because $t = o(N)$. Therefore $I = \bigcap_{i=1}^{t} JK(P_i)K(G)$. Now $\hat{P}_i$ annihilates $JK(P_i)K(G)$ on the left so $\hat{P}_i I = 0$ $\forall i$. Since $\hat{G} = \hat{N}\hat{P}$ we have $\hat{G}I = 0$. Now $G = N \cup P_1 \cup P_2 \cup \cdots \cup P_t$ where unions are disjoint on the nonidentity elements. Therefore $\hat{G} + t \cdot 1 = \hat{N} + \Sigma_{i=1}^{t} \hat{P}_i$ in $K(G)$. But $\hat{P}_i I = 0$ $\forall i$ and $\hat{G}I = 0$. Therefore $(t \cdot 1 - \hat{N})I = 0$. Let $\alpha \in I$. Then $(t \cdot 1 - \hat{N})\alpha = 0$ and so $\alpha = \hat{N}\alpha/t$ (because $t \neq 0$ in K). Therefore $\alpha \in \hat{N}K(G)$. But $G = NP$ and $\hat{N}n = \hat{N}$ $\forall n \in N$. Therefore $\alpha \in \hat{N}K(P)$. Let $\alpha = \hat{N}\beta$ where $\beta \in K(P)$ and $\beta = \Sigma_{y \in P} a_y y$. Since $\alpha \in I \subset w(K(G))$ we have $t \cdot (\Sigma_{y \in P} a_y) = 0$ and so $\Sigma_{y \in P} a_y = 0$ (because $t \neq 0$ in K). Thus $\beta \in w(K(P)) = JK(P)$. Hence $\alpha \in \hat{N}JK(P)$. This proves Theorem 1.

COROLLARY 1. *Let G be a finite group and $p \mid o(G)$. Suppose there exists a normal subgroup G_0 of G such that $p \nmid (G : G_0)$ and G_0 is a Frobenius group with P as a complement subgroup. Then $JK(G) = \bigcap_{x \in G} JK(P^x)K(G)$.*

PROOF. $JK(G) = JK(G_0)K(G)$ [5, Theorem 16.6] and by Theorem 1, $JK(G_0) = \bigcap_{x \in G_0} JK(P^x)K(G_0)$. Hence

$$JK(G) = \left(\bigcap_{x \in G_0} JK(P^x)K(G_0) \right) K(G)$$
$$= \bigcap_{x \in G_0} JK(P^x)K(G) = \bigcap_{x \in G} JK(P^x)K(G).$$

(The last equality follows from the fact that the Sylow p-subgroups of G are precisely those of G_0.)

Professor Passman in a letter asked if the converse of Corollary 1 holds. He thus asked: If G is a finite group such that $p|o(G)$ and $JK(G) = \bigcap JK(P)K(G)$ as P ranges over Sylow p-subgroups of G for some field K of characteristic $p \neq 0$, does there exist in G a normal subgroup G_0 such that $p \nmid (G : G_0)$ and that G_0 is a Frobenius group with a Sylow p-subgroup as a complement? We give below a generalization (Corollary 2) of Corollary 1. This generalization yields a negative answer to the above question.

COROLLARY 2. *Suppose there exist subgroups G_0, P_0 of the finite group G such that $G \supset G_0 \underset{\neq}{\supset} P \underset{\neq}{\supset} P_0$, $G_0 \lhd G$, $P_0 \lhd G_0$ and G_0/P_0 is a Frobenius group with P/P_0 as a complement. Then $JK(G) = \bigcap JK(P)K(G)$, P ranging over Sylow p-subgroups of G.*

PROOF. Since G_0/P_0 is a Frobenius group with P/P_0 as a complement. By Theorem 1 we have

$$JK(G_0/P_0) = \bigcap_{x \in G_0} w(K(P^x/P_0))K(G_0/P_0).$$

Taking the complete preimage of both sides under the canonical map from $K(G_0)$ to $K(G_0/P_0)$ we get

$$JK(G_0) = \bigcap_{x \in G_0} JK(P^x)K(G_0).$$

$JK(G) = JK(G_0)K(G)$ [**5**, Theorem 16.6]. Hence

$$JK(G) = \left(\bigcap_{x \in G_0} JK(P^x)K(G_0) \right) K(G)$$
$$= \bigcap_{x \in G_0} JK(P^x)K(G) = \bigcap_{x \in G} JK(P^x)K(G).$$

EXAMPLE. For $p = 2$, S_4 satisfies the conditions of Corollary 2, by taking $G_0 = S_4$ and $P_0 = V_4$, but S_4 does not satisfy the conditions of Corollary 1. This answers the question of Passman. Of course, now one can ask: Is the converse of Corollary 2 true?

3. Counterexample. The equality $JK(G) = \bigcap JK(P)K(G)$ where P ranges over p-Sylow subgroups of G is false in general, as the following example shows.

EXAMPLE (PASSMAN). Let

$$N = \left\{ \begin{bmatrix} 1 & \alpha & \beta \\ 0 & 1 & \gamma \\ 0 & 0 & 1 \end{bmatrix} \middle| \alpha, \beta, \gamma \in Z/3Z \right\}.$$

N is a group of order 27 and $\forall x \in N$, $x^3 = 1$. It is easy to see that N is generated by a, b, c where

$$a = \begin{bmatrix} 1 & 1 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}, \quad b = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{bmatrix} \quad \text{and} \quad c = \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix};$$

$(a, b) = c$ and c is central. Define $\sigma: N \to N$ by

$$\sigma \begin{bmatrix} 1 & \alpha & \beta \\ 0 & 1 & \gamma \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 1 & -\alpha & \beta \\ 0 & 1 & -\gamma \\ 0 & 0 & 1 \end{bmatrix}.$$

One checks that σ is an automorphism of N of order 2 and $\sigma(a) = a^2$; $\sigma(b) = b^2$; $\sigma(c) = c$. Let $H = \langle x \rangle$ the cyclic group of order 2.

Let $G = NX_\sigma H$ the semidirect product of N by H i.e. $xn = \sigma(n)x$ $\forall n \in N$. G is generated by the elements x, $a^{-1}xa$ and $b^{-1}xb$. So G is generated by the elements of order 2. $o(G) = 54$ and H is a Sylow 2-subgroup of G. Let K be an algebraically closed field of characteristic 2. We shall show that $JK(G) \neq \bigcap_{x \in G} JK(H^x)K(G)$. Since $G/N \approx H$ we have $G' \subset N$. Also $c = (a, b) \in G'$, $xa = a^2x = a(ax)$ so $a \in G'$, and $xb = b^2x = b(bx)$ so $b \in G'$. Thus $G' = N$. Hence $G = G'H$. Since $c \in G - H$ and c is central we have $N_G(H) \neq H$. So $G'H$ is not a Frobenius group with H as a complement. Hence by [7, Theorem, p. 103] $JK(G)$ is not central. But we shall show that $\bigcap_{x \in G} JK(H^x)K(G)$ is central. Let $I = \bigcap_{x \in G} JK(H^x)K(G)$. Now let x be an element of order 2 so that $P = \langle x \rangle$ is a Sylow 2-subgroup of G. Then

$$I \subset JK(P)K(G) = (1 + x)K(G)$$

so $(1 - x)I = 0$. Since G is generated by elements of order 2 therefore $w(K(G))$ is generated as left ideal by the set $\{(1 - x) | o(x) = 2\}$ [3, Lemma 1, p. 153]. Thus $wK(G)I = 0$ so $I \subset \hat{G}K(G)$ [3, Lemma 2, p. 154]. Hence I is central.

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