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MONOMIAL SPACE CURVES IN A^3 AS SET-THEORETIC COMPLETE INTERSECTIONS

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ABSTRACT. It is shown constructively that all monomial space curves in affine 3-space are set-theoretic complete intersections.

It was shown by J. Herzog (private communication) that all space curves in affine 3-space A^3 over an arbitrary field K, given parametrically by $x_1 = t^\alpha$, $x_2 = t^\beta$, $x_3 = t^\gamma$, α, β, γ positive integers, g.c.d.$(\alpha, \beta, \gamma) = 1$, are complete set-theoretic intersections. The following independent proof provides an algorithm to determine the surfaces involved explicitly and uses only "high-school algebra methods."

It is clear that we only have to consider space curves of the indicated type, which are not ideal-theoretic complete intersections. By [1], the prime ideal $P \subseteq K[x_1, x_2, x_3]$, defining such a curve C is given by

$$P = (f_1 = x_1^{\alpha_1} - x_2^{\alpha_{12}}x_3^{\alpha_{13}}, f_2 = x_2^{\alpha_2} - x_1^{\alpha_{21}}x_3^{\alpha_{23}}, f_3 = x_3^{\alpha_3} - x_1^{\alpha_{31}}x_2^{\alpha_{32}}),$$

where all exponents are integral, greater than 0 and satisfy the relations $\alpha_1 = \alpha_{21} + \alpha_{31}$, $\alpha_2 = \alpha_{12} + \alpha_{32}$, $\alpha_3 = \alpha_{13} + \alpha_{23}$. We claim:

$$D = (f_1, f_2, f_3) \cap (x_1^{\alpha_{21}}, x_2^{\alpha_{12}}) = (f_1, f_2, x_1^{\alpha_{21}}f_3, x_2^{\alpha_{12}}f_3) = (f_1, f_2).$$

PROOF. *1st equality*: $\supseteq$ is clear. If $f = \sum_{i=1}^3 g_i f_i \in D$, then $g_3 f_3 \in (x_1^{\alpha_{21}}, x_2^{\alpha_{12}})$. By [2], $(x_1^{\alpha_{21}}, x_2^{\alpha_{12}})$ is irreducible, hence primary. Since $f_3 \notin (x_1, x_2) = \sqrt{(x_1^{\alpha_{21}}, x_2^{\alpha_{12}})}$, $g_3 \in (x_1^{\alpha_{21}}, x_2^{\alpha_{12}})$, from which $\subseteq$.

2nd equality: $\supseteq$ is trivial. An easy calculation shows

$$x_1^{\alpha_{21}}f_3 = -x_2^{\alpha_{32}}f_1 - x_3^{\alpha_{13}}f_2, \qquad x_2^{\alpha_{12}}f_3 = -x_1^{\alpha_{31}}f_2 - x_3^{\alpha_{23}}f_1,$$

from which $\subseteq$. Therefore C and the line l with equations $x_1 = 0$, $x_2 = 0$ are the zeroes of (f_1, f_2). Since $C \cap l = \{(0, 0, 0)\}$, it suffices to construct a polynomial $g \in P$ such that $f_2^\mu \in (g, f_1)$, $g = x_3^\rho + h$, $h \in (x_1, x_2)$. To accomplish this we take $(x_2^{\alpha_2} - x_1^{\alpha_{21}}x_2^{\alpha_{23}})^{\alpha_1} = x_2^{\alpha_2}k \pm x_1^{\alpha_1\alpha_{21}}x_3^{\alpha_1\alpha_{23}}$, subtract or add $x_1^{\alpha_1(\alpha_{21}-1)}x_3^{\alpha_1\alpha_{23}}f_1$ and divide by $x_2^{\alpha_{12}}$. This gives $x_2^{\alpha_2-\alpha_{12}}k \pm x_1^{\alpha_1(\alpha_{21}-1)}x_3^{\alpha_{13}+\alpha_1\alpha_{23}} \in P$. If $\alpha_{21} = 1$ we are done; if not we show that the process, after proper modification, can be carried through α_{21}-times. To this end, consider the term $x_2^{j\alpha_2}x_1^{(\alpha_1-j)\alpha_{21}}x_3^{(\alpha_1-j)\alpha_{23}}$, $1 \leqslant j \leqslant \alpha_{21} - 1$, obtained from the binomial expansion of $f_2^{\alpha_1}$. Since $(\alpha_{21} - j)\alpha_1 < \alpha_{21}\alpha_1 - j\alpha_{21}$, this term can be changed by subtracting proper multiples of f_1 into $x_2^{j\alpha_2+(\alpha_{21}-j)\alpha_{12}}x_1^{(\alpha_1-j)\alpha_{21}-(\alpha_{21}-j)\alpha_1}x_3^{(\alpha_1-j)\alpha_{23}+(\alpha_{21}-j)\alpha_{13}}$

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and this term is divisible by $x_2^{\alpha_{12}}$ α_{21}-times. Q.E.D.

EXAMPLE. Let $x_1 = t^5$, $x_2 = t^6$, $x_3 = t^8$, $f_1 = x_1^4 - x_2^2x_3$, $f_2 = x_2^3 - x_1^2x_3$, $f_3 = x_3^2 - x_1^2x_2$. Subtracting proper multiples of f_1 from f_2^4 and dividing by x_2^2 gives:

$$\begin{array}{rr}
(x_2^3 - x_1^2x_3)^4 = & x_2^{12} - 4x_2^9x_1^2x_3 + 6x_2^6x_1^4x_3^2 - 4x_2^3x_1^6x_3^3 + x_1^8x_3^4 \\
& + x_2^2x_1^4x_3^5 - x_1^8x_3^4 \\
\hline
& x_2^{10} - 4x_2^7x_1^2x_3 + 6x_2^4x_1^4x_3^2 - 4x_2x_1^6x_3^3 + x_1^4x_3^5 \\
& + 4x_2x_1^6x_3^3 - 4x_2^3x_1^2x_3^4 \\
\hline
& x_2^{10} - 4x_2^7x_1^2x_3 + 6x_2^4x_1^4x_3^2 - 4x_2^3x_1^2x_3^4 + x_1^4x_3^5 \\
& + x_2^2x_3^6 - x_1^4x_3^5 \\
\hline
& x_2^8 - 4x_2^5x_1^2x_3 + 6x_2^2x_1^4x_3^2 - 4x_1^2x_2x_3^4 + x_3^6 = g.
\end{array}$$

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