ON THE NORM OF THE CANONICAL PROJECTION OF $E^{***}$ ONTO $E$ \perp

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Abstract. For each number $t$ with $1 < t < 2$, a renorming of $c_0$ is given for which $\|I - P\| = t$, where $P$ is the canonical projection of $c_0^{***}$ on $c_0^*$.

If $E$ is a Banach space, then its dual $E^*$ resides canonically in the third dual $E^{***}$. The mapping $P: E^{***} \to E^*$ that restricts a functional in $E^{***}$ to $E$ defines a projection of $E^{***}$ onto $E^*$ with norm one. Thus, if $E$ is nonreflexive of course, $1 < \|I - P\| < 2$. It is not hard to prove that if $E = c_0$, then $\|I - P\| = 1$. The paper \cite{1} is devoted to proving $\|I - P\| = 2$ for $E = l_1$. Furthermore, it seems that $\|I - P\|$ is either 1 or 2 in the (very few) examples in which the problem seems tractable. Fran Sullivan thus posed the question whether there are examples for which $1 < \|I - P\| < 2$. We answer this by constructing, for each number $t$ with $1 < t < 2$, a renorming of $c_0$ so that $\|I - P\| = t$.

We proceed as follows:

Let $0 < r < 1$. For $x = (x_1, x_2, \ldots) \in c_0$, define

\[ \|x\| = \max(r^{-1}|x_1|, \sup_n |x_n - x_1|). \]

Then we have the following:

1. $(1 + r)^{-1}\|x\|_{\infty} \leq \|x\| \leq \max(2, r^{-1})\|x\|_{\infty}$, where $\|x\|_{\infty} = \sup_n |x_n|$.

2. By virtue of (1), the dual of $c_0$ is $l_1$ and for $\lambda = (\lambda_1, \lambda_2, \ldots) \in l_1$ the dual norm is

\[ \|\lambda\| = \left( \sum_{n=1}^{\infty} \lambda_n \right) + \sum_{n=2}^{\infty} |\lambda_n|. \]

3. The second dual norm on $l_\infty$ is given by the same formula as $(\ast)$.\textsuperscript{(1)}

4. If $P$ is the canonical projection of $l_\infty^*$ onto $l_1$, then $\|I - P\| = 1 + r$.

Proofs. (1) If $\|x\| < 1$, then $|x_1| < r$ and $|x_n - x_1| < 1$. Hence, $|x_n| < 1 + r$ for $n > 1$. The other half is easier.

(2) Notice that for $x \in c_0$ we have

\[ \sum_{n=1}^{\infty} \lambda_n x_n = \left( r \sum_{n=1}^{\infty} \lambda_n \right) \left( \frac{1}{r} x_1 \right) + \sum_{n=2}^{\infty} \lambda_n (x_n - x_1) \]

\[ \leq \left( \left| \sum_{n=1}^{\infty} \lambda_n \right| + \sum_{n=2}^{\infty} \lambda_n \|x\| \right). \]

Hence the dual norm $(\ast)$ on $l_1$ is no larger than the norm (2). To see that

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we get equality, assume (without loss of generality) $\sum_{n=1}^{\infty} \lambda_n > 0$. Given $n$, let $x_j = 0$ for $j > n$; $x_1 = r$; $x_j = r + 1$ if $1 < j < n$ and $\lambda_j > 0$; and $x_j = r - 1$ if $1 < j < n$ and $\lambda_j < 0$. Then

$$\sum_{j=1}^{\infty} \lambda_j x_j = r \sum_{j=1}^{\infty} \lambda_j + \sum_{j=2}^{\infty} |\lambda_j|.$$ 

Hence the norm in (2) is the dual of norm (•).

(3) This can be obtained using the same formula,

$$\sum_{n=1}^{\infty} \lambda_n x_n = \left( r \sum_{n=1}^{\infty} \lambda_n \right) \left( \frac{1}{r} x_1 \right) + \sum_{n=2}^{\infty} \lambda_n (x_n - x_1),$$

as in (2). The details are omitted.

(4) We begin by defining $\mathcal{Q}_n : l_\infty \to l_\infty$ by letting the first $n$ coordinates of $\mathcal{Q}_n x$ be zero and for $j > n$ let the $j$th coordinate be $x_j$. Now, $||x|| < 1$ implies $|x_1| < r$ and $|x_j - x_1| < 1$, so $|x_j| < 1 + r$ for $j > 1$. Hence $||\mathcal{Q}_n x||_\infty < 1 + r$. But if the first term of an element of $l_\infty$ is zero, then its norm is the usual sup-norm. Thus $|| \mathcal{Q}_n || < 1 + r$.

Now we use the fact that $l_{\infty}^* \pi \pi$ can be considered as the space of Radon measures on the Stone-Cech compactification $\beta N$ of the positive integers. (We use $\mu$ interchangeably as a measure and linear functional.) The canonical projection $P$ of $l_{\infty}^*$ onto $l_\pi$ sends $\mu$ to the sequence $\{ \mu(n) \}$ and so $(I - P)\mu$ is the restriction of $\mu$ to $\beta N \setminus N$. Let $\lambda = P\mu$ and $\bar{\mu} = (I - P)\mu$. Since $\bar{\mu}(N) = 0$, it follows that $\bar{\mu}(\mathcal{Q}_n x) = \bar{\mu}(x)$ for each $n$ and each $x \in l_\infty$. Thus

$$|\bar{\mu}(x)| = |\bar{\mu}(\mathcal{Q}_n x)| = |\mu(\mathcal{Q}_n x) - \lambda(\mathcal{Q}_n x)|$$

$$\leq ||\mu|| ||\mathcal{Q}_n x|| + \sum_{j=n}^{\infty} |\lambda_j x_j|$$

$$\leq ||\mu||(1 + r)||x|| + ||x||_\infty \sum_{j=n}^{\infty} |\lambda_j|.$$ 

Taking the limit on $n$, we get $||\bar{\mu}|| \leq (1 + r)||\mu||$. Hence $||I - P|| \leq 1 + r$.

Now, let $s \in \beta N \setminus N$. Let $\epsilon_s$ be the point mass at $s$ and $\epsilon_1$ be the point mass at $1 \in N$. Take a net $\{n_\gamma\}$ in $N$ converging to $s$. If $||x|| < 1$, then $|x_{n_\gamma}| < 1 + r$ for each $n_\gamma > 1$, so

$$|\epsilon_s(x)| = \lim_{\gamma} |x_{n_\gamma}| < 1 + r.$$ 

Also $|x_{n_\gamma} - x_1| < 1$ for every $\gamma$ and therefore

$$|(\epsilon_s - \epsilon_1)(x)| = \lim_{\gamma} |x_{n_\gamma} - x_1| < 1.$$ 

Thus $||\epsilon_s|| < 1 + r$ and $||\epsilon_s - \epsilon_1|| < 1$. But if $y = (r, 1 + r, 1 + r, \ldots)$, then $||y|| = 1$, $\epsilon_s(y) = 1 + r$ and $|\epsilon_s - \epsilon_1(y)| = 1$. Hence $||\epsilon_s|| = 1 + r$, $||\epsilon_s - \epsilon_1|| = 1$, and $(I - P)(\epsilon_s - \epsilon_1) = \epsilon_s$ so $||I - P|| = 1 + r$. This completes the proof.

A. L. Brown has very recently communicated to us another way of renorming $c_0$ to achieve the results given above as follows: Let $T$ be an
isomorphism of $c_0$ onto the space of convergent sequences $c$. For each $t \in [0, 1]$, $\|x\|_t = t\|x\|_\infty + (1 - t)\|Tx\|_\infty$ defines an equivalent norm on $c_0$, and as $t$ ranges from 0 to 1 $\|I - P\|$ ranges from 2 to 1.

The referee has pointed out that all renormings of $c_0$ presented here have duals isometric to $I_1$. Thus $\|I - P\|$ is not an isometric invariant of $E^*$. He also points out that examples isometric to these were introduced for another purpose in [2].

REFERENCES


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