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THE CARDINALITY OF QUASICONFORMALLY NONEQUIVALENT TOPOLOGICAL 3-BALLS WITH FLAT BOUNDARIES IS c

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ABSTRACT. The theorem mentioned in the title is proved. During the course of the proof, the failure for $n = 3$ of the following 2-dimensional result will also be established: The boundary of a Jordan domain D in n-space is a quasiconformal $(n - 1)$-sphere if every quasiconformal self-mapping of D can be extended to a quasiconformal self-mapping of the whole space.

1. Introduction. Let Σ^{n-1} be a topological $(n - 1)$-sphere imbedded in the n-sphere S^n. By the Jordan-Brouwer separation theorem, Σ^{n-1} divides S^n into two domains, D_1 and D_2, and is their common boundary. The set Σ^{n-1} is *collared* if there is a neighborhood U of Σ^{n-1} and a homeomorphism h of $U \cap \bar{D}_1$ (or of $U \cap \bar{D}_2$) into S^n carrying Σ^{n-1} onto the equator S^{n-1} of S^n. The set Σ^{n-1} is *bicollared* if h is defined in all of U and maps Σ^{n-1} onto S^{n-1}. The set Σ^{n-1} is *flat* if there is a homeomorphism h of S^n onto itself which carries Σ^{n-1} onto S^{n-1}. By results of Brown **[1]**, **[2]**, every bicollared Σ^{n-1} in S^n is flat and a collared component of $S^n - \Sigma^{n-1}$ is a topological n-ball.

The set Σ^{n-1} is said to be *quasiconformally collared* (resp. *quasiconformally bicollared*) if the homeomorphism h above is quasiconformal. The image of S^{n-1} under a quasiconformal mapping of S^n is generally referred to as a *quasiconformal sphere*, rather than a quasiconformally flat sphere. Gehring **[4]** has established quasiconformal analogues to Brown's results. In particular, a quasiconformally collared component of $S^n - \Sigma^{n-1}$ is a quasiconformal n-ball. The other component of $S^n - \Sigma^{n-1}$ need not be a quasiconformal n-ball, even in the case that Σ^{n-1} is flat.

Let $\mathfrak{D}$ be the collection of all topological n-balls in S^n whose boundaries are flat $(n - 1)$-spheres and whose exteriors are quasiconformal n-balls. We divide $\mathfrak{D}$ into equivalence classes by regarding two domains in $\mathfrak{D}$ as equivalent if they can be mapped quasiconformally onto each other. Let $E(\mathfrak{D})$ denote the set of equivalence classes so obtained. We show that in 3-space $E(\mathfrak{D})$ has the cardinality of a continuum. This stands in marked contrast with the situation in 2-space, where the corresponding cardinality is well known to be one. (For related questions, see Kopylov **[6]**.) In the course of the proof, the failure for $n = 3$ of the following result, due to Rickman **[8]**

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for $n = 2$, will also be established: The boundary of a Jordan domain D in n-space is a quasiconformal $(n-1)$-sphere if every quasiconformal self-mapping of D can be extended to a quasiconformal self-mapping of the whole space.

2. Wedges. We consider domains D in $\overline{R}^3 = R^3 \cup \{\infty\}$,

$$D = \{x = (x_1, x_2, x_3) \in R^3 \colon |x_2| < g(x_1), x_1 > 0\}, \tag{1}$$

where the function $g\colon [0, \infty) \to R^1$ satisfies the following conditions for some $0 < a < \infty$:

$$\begin{cases} \text{(i)} & g \text{ is continuous, } g(0) = 0, g(u) > 0 \text{ for } u > 0, \\ & \text{and } g(u) = g(a) \text{ for } u \geqslant a. \\ \text{(ii)} & g' \text{ is continuous, bounded, and increasing in} \\ & (0, a). \\ \text{(iii)} & \lim_{u \to 0} g'(u) = 0. \end{cases} \tag{2}$$

Such a domain D is called a *wedge* of angle zero. The union of the x_3-axis and the point ∞ is called the *edge* of D. (The above terminology is taken from Gehring and Väisälä **[5]**.) Obviously a wedge D is a Jordan domain whose boundary ∂D is a flat 2-sphere. The exterior of D is a quasiconformal 3-ball, while D is not. (See Gehring and Väisälä **[5]**.) Hence ∂D is not a quasiconformal 2-sphere, i.e. ∂D is not quasiconformally bicollared.

We will show that no two of the wedges defined by the functions $g(u) = u^p$, $p \in (1, \infty)$, can be mapped quasiconformally onto one another. For this we require an upper and a lower bound for the modulus $M(\Gamma)$ of a certain path family Γ. We let $F(\Gamma)$ denote the set of all Borel-measurable extended real-valued functions ρ in R^3 for which

$$\int_\gamma \rho \, ds \geqslant 1$$

for each locally rectifiable path $\gamma \in \Gamma$. The modulus of Γ is defined as

$$M(\Gamma) = \inf_{\rho \in F(\Gamma)} \int_{R^3} \rho^3 \, dm.$$

(For the theory of modulus and quasiconformal mappings, see Gehring **[3]** and Väisälä **[10]**.)

Lemma 1. *Let D be the wedge defined by the function $g(u) = u^p$ $(p > 1)$, let $r_0 > 0$ be a number such that $0 < g'(r_0) \leqslant 1$, for $0 < r \leqslant r_0$ let*

$$Z(r) = \{x = (x_1, x_2, x_3) \colon x_1^2 + x_3^2 < r\},$$

and for $0 < r_1 < r_2 \leqslant r_0$ let $\Gamma(r_1, r_2)$ denote the family of all paths joining $\partial Z(r_1)$ and $\partial Z(r_2)$ in $D \cap Z(r_2) - \overline{Z(r_1)}$. Then

$$\frac{A(p)}{\left(r_1^{(1-p)/2} - r_2^{(1-p)/2}\right)^2} \leqslant M(\Gamma(r_1, r_2)) \leqslant \frac{2^{3/2}A(p)}{\left(r_1^{(1-p)/2} - r_2^{(1-p)/2}\right)^2},$$

where

$$A(p) = 2^{-5/2}(p-1)^2 \int_0^{\pi} (\sin\varphi)^p \, d\varphi. \tag{3}$$

PROOF. For the left-hand inequality, let $\rho \in F(\Gamma(r_1, r_2))$, let (r, φ, x_2) be cylindrical coordinates in R^3 with the polar angle φ being measured from the positive half of the x_3-axis, and for $r \in [r_1, r_2]$, $\varphi \in (0, \pi)$, $v \in (-1, 1)$ let

$$\gamma_{\varphi v}(r) = (r, \varphi, vg(r \sin\varphi)).$$

Since $\gamma_{\varphi v}$ is a rectifiable path in $\Gamma(r_1, r_2)$ and since $g'(r_2) \leqslant 1$ by hypothesis and by (2), we obtain

$$\begin{aligned} 1 &\leqslant \left(\int_{\gamma_{\varphi v}} \rho \, ds\right)^3 \leqslant \left(2^{1/2}\int_{r_1}^{r_2} \rho \, dr\right)^3 \\ &\leqslant 2^{3/2}\int_{r_1}^{r_2} \rho^3 \, rg(r \sin\varphi) dr \left(\int_{r_1}^{r_2} r^{-1/2} g(r\sin\varphi)^{-1/2} dr\right)^2 \end{aligned}$$

by Hölder's inequality. Integrating with respect to φ and v yields

$$\begin{aligned} \int_{R^3} \rho^3 \, dm &\geqslant \int_{-1}^{1} dv \int_0^{\pi} d\varphi \int_{r_1}^{r_2} \rho^3 \, rg(r\sin\varphi) dr \\ &\geqslant A(p)\left(r_1^{(1-p)/2} - r_2^{(1-p)/2}\right)^{-2}, \end{aligned}$$

where $A(p)$ is as in (3). Since $\rho \in F(\Gamma(r_1, r_2))$ was arbitrary, this gives the left-hand inequality.

The right-hand inequality is obtained by observing that

$$\rho(x) = \begin{cases} \dfrac{p-1}{2r^{(p+1)/2}\left(r_1^{(1-p)/2} - r_2^{(1-p)/2}\right)} \\ \qquad\qquad \text{if } x = (r, \varphi, x_2) \in D \cap Z(r_2) - \overline{Z(r_1)}, \\ 0 \quad \text{otherwise,} \end{cases}$$

belongs to $F(\Gamma(r_1, r_2))$.

We also need the following extension result which shows, in particular, that, contrary to the situation in the plane (Rickman [**8**]), the extendability of 3-dimensional quasiconformal mappings over a flat 2-sphere does not guarantee that the 2-sphere will be quasiconformal.

LEMMA 2. *All quasiconformal mappings between wedges can be extended to quasiconformal self-mappings of $\overline{R}^3$.*

PROOF. This result was proved in [7].

LEMMA 3. *Let D and D^* be two wedges defined respectively by the functions $g(u) = u^p$ and $g^*(u) = u^{p^*}$, $p, p^* \in (1, \infty)$. Then D can be mapped quasiconformally onto D^* if and only if $p = p^*$.*

PROOF. The sufficiency part is obvious. For the necessity part, suppose, for example, that $p < p^*$, and that, contrary to the assertion, there is a quasiconformal mapping f of D onto D^*. By Lemma 2, f can be extended to a quasiconformal mapping of $\bar{R}^3$ onto itself. Denote this mapping again by f, let E denote the common edge of D and D^*, and for $x \in E - \{\infty, f(\infty)\}$ set

$$L(x, f^{-1}) = \limsup_{h \to 0} \frac{|f^{-1}(x+h) - f^{-1}(x)|}{|h|}.$$

In the proof of Lemma 2 it is verified that $f(E) = E$. Utilizing an idea of Syčev [9], we note the existence of a point x_0 in $E - \{\infty, f(\infty)\}$ such that

$$L(x_0, f^{-1}) > 0.$$

Otherwise f^{-1} would be locally constant in $E - \{\infty, f(\infty)\}$. Assume, for convenience of notation, that $x_0 = 0 = f^{-1}(x_0)$. Let

$$L(r) = \max_{|x|=r} |f(x)|, \quad l(r) = \min_{|x|=r} |f(x)|,$$

$$H = \limsup_{r \to 0} \frac{L(r)}{l(r)}.$$

Since $H < \infty$ by the quasiconformality of f, there exist positive constants r_0 and H_0 with $g'(r_0) \in (0, 1]$ such that

$$L(r)/l(r) \leqslant H_0 \tag{4}$$

whenever $r \in (0, r_0]$. Choose r_0^* so that $g^{*\prime}(r_0^*) \in (0, 1]$ and $D^* \cap Z(r_0^*)$ lies in $f(D \cap Z(r_0))$, where, for $r \in (0, 1]$, $Z(r)$ is as defined in Lemma 1. Next choose $c \in (0, L(0, f^{-1}))$ and let (x_k) be a sequence of points in D such that $x_k \to 0$, $|x_k| < r_0$, $|f(x_k)| < r_0^*$, and

$$|x_k|/|f(x_k)| \geqslant c. \tag{5}$$

Denoting $|x_k| = r_k$ and using (4) and (5) we obtain

$$L(r_k) \leqslant H_0 l(r_k) \leqslant H_0 |f(x_k)| \leqslant H_0 r_k / c = C_0 r_k, \tag{6}$$

where $C_0 = H_0/c$. Passing to a subsequence, we may assume that $C_0 r_k < r_0^*$ for every k. Since $g'(r_0) \leqslant 1$, it follows from (6) that $f(D \cap Z(r_k/2))$ lies in $D^* \cap Z(C_0 r_k)$. Let $\Gamma(r_k/2, r_0)$ be the family of all paths joining $\partial Z(r_k/2)$ and $\partial Z(r_0)$ in $D \cap Z(r_0) - \overline{Z(r_k/2)}$ and let $\Gamma^*(C_0 r_k, r_0^*)$ be the family of all paths joining $\partial Z(C_0 r_k)$ and $\partial Z(r_0^*)$ in $D^* \cap Z(r_0) - \overline{Z(C_0 r_k)}$. Since $f\Gamma(r_k/2, r_0)$ is minorized by $\Gamma^*(C_0 r_k, r_0^*)$, we obtain

$$\begin{aligned}\frac{M(\Gamma(r_k/2, r_0))}{M(f\Gamma(r_k/2, r_0))} &\geqslant \frac{M(\Gamma(r_k/2, r_0))}{M(\Gamma^*(C_0 r_k, r_0^*))}\\ &\geqslant 2^{-3/2}\frac{A(p)\left[(r_k/2)^{(1-p)/2} - r_0^{(1-p)/2}\right]^{-2}}{A(p)\left[(C_0 r_k)^{(1-p^*)/2} - r_0^{*(1-p^*)/2}\right]^{-2}}\\ &\geqslant C^* r_k^{p-p^*}\end{aligned}$$

by Lemma 1, where C^* is a positive constant which does not depend on k. Letting $k \to \infty$ leads to a contradiction with the quasiconformality of f. The proof is complete.

3. Results. Since the boundary of a wedge is not a quasiconformal sphere, Lemma 2 yields:

THEOREM 1. *In 3-space there are Jordan domains D whose boundaries are flat, but not quasiconformally flat, such that all quasiconformal self-mappings of D can be extended to quasiconformal self-mappings of $\overline{R}^3$.*

Since the cardinality of the collection of all subdomains of $\overline{R}^3$ is **c**, the cardinality of a continuum, Lemma 3 yields:

THEOREM 2. *In 3-space the cardinality of a maximal collection of quasiconformally nonequivalent Jordan domains with flat boundaries and quasiconformally collared exteriors is* **c**.

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