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# UNIQUE ERGODICITY FOR CERTAIN RANDOM TRANSLATIONS

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ABSTRACT. Spatially dependent convex combinations of a pair of irrationally related translations on $\mathbf{R}$ are shown to admit at most one invariant probability. The only condition on the coefficient functions is measurability and essential positivity.

A particle moves on $(-\infty, \infty)$ at times $t = 0, 1, \ldots$. If it is at $x$ at time $t = n$, it moves to $x + a$ with probability $p(x)$ and to $x - b$ with probability $q(x) = 1 - p(x)$. We assume that both $a$ and $b$ are positive, so that the particle moves back and forth. We want to avoid the case where the possible positions of the particle are contained in a translate of a discrete subgroup of $(-\infty, \infty)$, so we assume that $a/b$ is irrational. We also assume, of course, that the function $p$ is Borel measurable on $(-\infty, \infty)$ into $[0, 1]$. Consider the operator $P$ defined for bounded Borel-measurable functions $f$ on $(-\infty, \infty)$ by

$$Pf(x) = f(x + a)p(x) + f(x - b)q(x), \qquad x \in (-\infty, \infty).$$

This is a Markov operator corresponding to the transition probability function $P(x, B) = P1_B(x)$ for $x \in (-\infty, \infty)$ and $B$ in the class $\mathfrak{B}$ of Borel subsets of $(-\infty, \infty)$. A probability measure $\mu$ on $\mathfrak{B}$ is said to be *invariant* if $\mu(B) = \int \mu(dx)P(x, B)$ for all $B \in \mathfrak{B}$. The following is an extension of a result due to Frank Norman and J. W. Pickands III [1, p. 219, Theorem 4.2].

THEOREM. *If $a/b$ is irrational and $0 < p < 1$, then there is at most one invariant probability measure.*

PROOF. Assume that $a/b$ is irrational and that $0 < p < 1$. We shall show first that any invariant probability is Lebesgue continuous. To this end, let $\mu$ be an invariant probability on $(-\infty, \infty)$. By a harmless rescaling we arrange that $a + b = 1$. Let $\{X_n: n \geq 1\}$ be the Markov process on $(-\infty, \infty)$ with initial distribution $\mu$ and transition probability $p(x, E)$. Since $\mu$ is invariant, $\{X_n: n \geq 1\}$ is stationary. Let $f(x) = \exp(2\pi ix)$ for $x$ in $(-\infty, \infty)$. Then $f$ is a map of $(-\infty, \infty)$ onto the 1-torus $T$ which is certainly measurable relative to the Borel subsets $\mathfrak{B}(T)$ of $T$. Let $Z_n = f(X_n)$. Since $a + b = 1$, $\{Z_n: n \geq 1\}$ is a deterministic Markov process with $Z_{n+1} = Z_n e^{2\pi ia}$. The $\{Z_n\}$

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process is stationary by virtue of the stationarity of the $\{X_n\}$ process so the initial distribution must be $\lambda$, the Lebesgue measure on $T$. Thus $\lambda(B) = \mu(f^{-1}(B))$ for all $B$ in $\mathfrak{B}(T)$. Let $n$ be any integer and suppose $A$ is a Borel subset of $[n, n + 1]$ with $\mu(A) > 0$. Then $\lambda(f(A)) = \mu(f^{-1}f(A)) \geqslant \mu(A) > 0$. But $\lambda(f(A)) = m(A)$ where $m$ denotes Lebesgue measure on $(-\infty, \infty)$. Since this holds for all $n$, $\mu(B) > 0$ implies $m(B) > 0$; that is, $\mu \ll m$ and so there is a Borel measurable $g \geqslant 0$ with $\mu(B) = \int_b g\, dm$ for all $B$ in $\mathfrak{B}(-\infty, \infty)$.

Suppose $\mu_1$ and $\mu_2$ are distinct invariant probabilities. Then the signed measure $\mu_1 - \mu_2$ is nonzero and invariant. The positive and negative parts of $\mu_1 - \mu_2$ are each invariant and are mutually singular. So if there is more than one invariant probability we can produce a pair of invariant Lebesgue continuous probabilities which are mutually singular. Thus we can obtain two measurable support sets $A$ and $B$ both of positive (though possibly infinite) Lebesgue measure. Now the support of an invariant probability $\mu$ has the property that $\mu$-a.e. point of supp $\mu$ remains in supp $\mu$ forever. Then we can pick $A'$ in $A$ to be of finite positive Lebesgue measure so that no point of $A'$ ever hits $B$. Thus $A' + ra - sb \cap B$ is empty for all positive integer pairs $(r, s)$. Let $f$ be an $L_1$ (Lebesgue) function strictly positive on $B$ and vanishing off of $B$. Then $1_{A'} * f$ is continuous and not identically zero. However this function vanishes on a translate of the dense set $(ra - sb\colon r, s \text{ in } Z_+)$ and the contradiction finishes the argument.

REMARKS. In [**1**, p. 217] and with greater generality in [**2**] Frank Norman gives sufficient conditions for the existence of invariant probabilities in terms of expected step size.

There is a generalization of this result to several terms and several dimensions (unfortunately simultaneously). The $n$-torus with a Kronecker irrational homeomorphism plays the role of the irrational rotation of $T$. We spare the reader the details.

## REFERENCES

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