OPEN NONNEGATIVELY CURVED 3-MANIFOLDS WITH A POINT OF POSITIVE CURVATURE

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ABSTRACT. Let $M$ be a complete open nonnegatively curved Riemannian 3-manifold with a point at which all sectional curvatures are positive, and suppose that $M$ contains a pole. Then $M$ is not flat on the complement of any compact set. Note that this is clearly false for 2-manifolds.

In this short note I will prove the following: Let $M^3$ be a complete open nonnegatively curved 3-manifold with a point at which all sectional curvatures are positive. Suppose further that $M$ contains a pole. Then $M$ is not flat off any compact set.

Despite the ease with which this is proven, I feel that this formal presentation is warranted on several grounds. Among others, these include the following: (1) the result is, perhaps, counterintuitive—it is clearly false in two dimensions; (2) it opens up similar questions in higher dimensions which cannot be so easily answered; (3) finally, if the need for a pole represents a defect in the proof and not a real restriction, which I believe to be the case, it would lead to one of the few instances in which a geometric condition at one point implies a global geometric (not topological) result.

Notation and preliminary remarks. (1) $M$ will always denote a complete Riemannian manifold.

(2) A point $p$ in $M$ is called a pole (see [2]) if the exponential map $\exp_p: M_p \to M$ is a submersion or has maximal rank on all $M_p$. If $p$ is a pole, $\exp_p$ will be a diffeomorphism if $M$ is simply connected.

(3) If $M^3$ is an open nonnegatively curved 3-manifold with a point of positive curvature, then (see [1]) $M$ is diffeomorphic to $\mathbb{R}^3$; in particular it is simply connected.

(4) $T_x(A)$ will denote the open tubular neighborhood of radius $r$ about the set $A \subset M$, and $S_r(A)$ will denote $\partial T_r(A)$, the sphere of radius $r$ about $A$. N.B. $S_r(A)$ is contained in $M$, not $TM$.

(5) If $x \in M$ and $\gamma$ is a normal geodesic ray in $M$ with $\gamma(0) = x$, then $H_x(\gamma)$ will denote the complementary half-space determined by $x$ and $\gamma$; i.e. $H_x(\gamma) = M \setminus \{ y \in M | d(\gamma(t), y) < t, \ t \in [0, \infty) \}$. Recall that $H_x(\gamma)$ is totally convex if $M$ is nonnegatively curved (see [1]).

(6) If $C$ is an oriented codimension one submanifold of $M$ with orientation vector field $N$, i.e., $N$ is a unit length section of the normal bundle of $C$ in...
Let $\Pi_N$ be the second fundamental form on $C$ relative to $N$, so that $\Pi_N(X, Y) = \langle \nabla_X N, Y \rangle$ for $X$ and $Y$ in $T C$.

(7) We call a $q$-form with values in $p$-forms a $(q, p)$-form. For a résumé of the calculus of $(q, p)$-forms, see for example [3].

It would seem that the following two lemmas, although surely known in some quarters, do not appear in the literature. Hence I have included brief proofs.

**Lemma 1.** Let $M$ be a complete open nonnegatively curved manifold, and let $p \in M$ be a pole. If $x \in S_r(p)$ and $\gamma$ is the ray originating at $p$ and passing through $x$, let $N(x) = \gamma'(x)$ determine an orienting vector field $N$ for $S_r(p)$. Then $\Pi_N$ is positive semidefinite on $S_r(p)$.

**Proof.** Let $x$ and $\gamma$ be as in the statement of the lemma, and suppose that $\gamma(0) = p$, $\gamma(a) = x$. Then $\gamma |_{(a, \infty)}$ is a ray originating at $x$, and so the complementary half space $H_x(\gamma)$ may be constructed. Clearly $H_x(p) \subset H_x(\gamma)$, and so the support plane for $H_x(\gamma)$ at $x$ is a support plane for $T_r(p)$ at $x$. Since $H_x(\gamma)$ is at least locally convex, the lemma follows. □

**Remark.** Using the same underlying idea one could easily give an elementary, if somewhat longer, proof of this lemma using only the Rauch comparison theorem.

**Lemma 2.** Let $M^n$ be a complete open nonnegatively curved manifold. Let $C \subset M$ be an oriented codimension one submanifold with orientation vector field $N$. Extend $N$ to a neighborhood of $C$ by unit speed geodesics; i.e., if $y = \exp_x t_0 N(x)$ for $x \in C$, let $N(y) = (\exp_x t N(x))'(t_0)$. Furthermore suppose that $\Pi_N$ is positive semidefinite on $C$. Set $C_t = \{ y \mid y = \exp_x t N(x) \text{ for some } x \in C \}$. Then $(d/dt)(\det \Pi_N)_{t_0} < 0$ as long as $C_t$ is smooth and $\Pi_N > 0$ on $C_t$ for all $0 < t < \rho$.

**Proof.** Regard $\Pi_N$ as a $(1, 1)$-form with $\Pi_N(X)(Y) = \langle \nabla_X N, \cdot \rangle(Y) = \langle \nabla_X N, Y \rangle$. Then we can write $\int_C \det \Pi_N = \int_C \langle N, \cdot \rangle \wedge \Pi_N^{-1}$, where $\langle N, \cdot \rangle$ is a $(0, 1)$-form, and thus $\langle N, \cdot \rangle \wedge \Pi_N^{-1}$ is an $((n - 1), n)$-form, and thus is integrable over $C_t$. Furthermore,

$$
\int_{C_t} \langle N, \cdot \rangle \wedge \Pi_N^{-1} - \int_{C_{t-\epsilon}} \langle N, \cdot \rangle \wedge \Pi_N^{-1} = \int_{D_{t,\epsilon}} d(\langle N, \cdot \rangle \wedge \Pi_N^{-1}),
$$

where $D_{t,\epsilon} = T_t(C) \setminus T_{t-\epsilon}(C)$. But an easy computation yields

$$
\int_{D_{t,\epsilon}} d(\langle N, \cdot \rangle \wedge \Pi_N^{-1}) = \int_{D_{t,\epsilon}} \Pi_N + (n - 1) \int_{D_{t,\epsilon}} \langle N, \cdot \rangle \wedge \Omega_N \wedge \Pi_N^{-2},
$$

where $\Omega_N$ is the $(2, 1)$-form $\Omega_N(X, Y)(Z) = \langle R(X, Y)N, \cdot \rangle(Z) = \langle R(X, Y)N, Z \rangle$.

Since $\nabla_N N = 0$, $\Pi_N = 0$ on $D_{t,\epsilon}$. Furthermore, if $X_1, \ldots, X_n$ is a local orthonormal basis of eigenvectors of $\Pi_N$ with eigenvalues $\lambda_1, \ldots, \lambda_n$, and $X_1 = N$, and if $K_{ij} = -\langle R(X_i, X_j)X_i, X_j \rangle$, then
\[ \langle N, \cdot \rangle \wedge \Omega_N \wedge \Pi_N^{n-2}(X_1, \ldots, X_n)(X_1, \ldots, X_n) = (n - 2)! \sum_{j=2}^{n} - \left[ K_{ij} \left( \prod_{i \neq j} \lambda_i \right) \right]. \]

But since \( K_{ij} \geq 0 \) and \( \lambda_i \geq 0 \) for all \( 0 < i, j < n \),

\[ \int_{C_r} \langle N, \cdot \rangle \wedge \Pi_N^{n-1} - \int_{C_{r-t}} \langle N, \cdot \rangle \wedge \Pi_N^{n-1} < 0. \]

**Proposition.** Let \( M^3 \) be a complete open nonnegatively curved 3-manifold with a point \( q \) in \( M \) at which all sectional curvatures are positive. Suppose further that \( M \) contains a pole \( p \). Then \( M \) is not flat off any compact set.

**Proof.** Let \( r \) be any real number and let \( X \) and \( Y \) denote vector fields over \( S_r(p) \). Let \( K_r(X, Y) \) denote the sectional curvature in \( M \) of the section spanned by \( X \) and \( Y \), and let \( \overline{K}_r(X, Y) \) denote the curvature in the induced metric in \( S_r(p) \) of the section spanned by \( X \) and \( Y \). Then it is standard that \( K_r(X, Y) = \overline{K}_r(X, Y) - \det \Pi(X, Y) \). Furthermore, the Gauss-Bonnet theorem states that \( \int_{S_r(p)} \overline{K}_r = 4\pi \), so that

\[ \int_{S_r(p)} K_r = 4\pi - \int_{S_r(p)} \det \Pi. \]

But Lemma 2 then implies that \( (d/dt)\int_{S_r(p)} K_r > 0 \). Hence if \( r > d(p, q) \), \( \int_{S_r(p)} K_r > 0. \]

**Remark.** Note that, since \( \exp_p \) is volume decreasing (\( K > 0 \)), one obtains a bound on the rate at which sectional curvature density can decrease as \( r \) increases.

**References**


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